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On the functional inequality $\left|f\left(\frac{x+y}{2}\right)\right| \leqq \frac{|f(x)|+|f(y)|}{2}$

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§1. Considering the Cauchy's functional equation

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}, \tag{1}$$

where $f(z)$ is an entire function of z, we have the following functional inequality:

$$\left|f\left(\frac{x+y}{2}\right)\right| \leqq \frac{|f(x)|+|f(y)|}{2}. \tag{2}$$

In this paper we shall determine all the entire functions $f(z)$ which satisfy (2).

THEOREM. *If $f(z)$ is an entire function of z, then all the functions which satisfy (2) are $(\alpha z+\beta)^n$ and $\exp(\alpha z+\beta)$ where α, β are arbitrary complex constants and n is an arbitrary natural number, and only these.*

PROOF. We may assume that $f(z)\not\equiv 0$. Putting $z=s+it$ (s, t real), $\varphi(s, t) =|f(z)|$ and using a real parameter τ[1], the function

$$F(\tau)=\varphi(a+h\tau, b+k\tau)+\varphi(a-h\tau, b-k\tau)$$

has a minimum $2\varphi(a, b)$ at $\tau=0$ by (2). Here a, b, h, k are arbitrary real constants which satisfy $f(a+ib)\neq 0$. Hence we have $F''(0)\geqq 0$. Since

$$F''(0)=2\{\varphi_{ss}(a, b)h^2+2\varphi_{st}(a, b)hk+\varphi_{tt}(a, b)k^2\},$$

we have

$$\varphi_{ss}(a, b)h^2+2\varphi_{st}(a, b)hk+\varphi_{tt}(a, b)k^2 \geqq 0.$$

Since h, k are arbitrary, we have

$$\varphi_{st}^2(a, b)-\varphi_{ss}(a, b)\varphi_{tt}(a, b)\leqq 0. \tag{3}$$

Since $f(a+ib)\neq 0$, there exists a regular branch $g(z)$ of $\sqrt{f(z)}$ in a properly chosen vicinity V of $z=\gamma=a+ib$.

Using the Cauchy-Riemann equations, we have

$$\{\varphi_{st}(a, b)\}^2-\varphi_{ss}(a, b)\varphi_{tt}(a, b)=4\{|g(\gamma)g''(\gamma)|^2-|g'(\gamma)|^4\}.$$

By (3) we have