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On the partial recursive functions of ordinal numbers

Dedicated to Professor Y. Akizuki for his 60th birthday

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On the basis of a series of Takeuti's papers [5], [6] and [7], G. Takeuti and A. Kino [8] developed the theory of recursive functions of ordinals, defined by proposing schemata, and that of hierarchy of predicates of ordinals, built on it, and they obtained various remarkable results. We are interested in giving a formalism for those functions and in applying it, via arithmetization, to the investigation of that hierarchy which contributes not only to the theory of ordinal numbers but also to the effective and classical descriptive set theory (cf., especially, §§ 7-9 of [8]).

In the meanwhile, M. Machover [4] presented a formal system of recursive functions of ordinal numbers with infinitely many variables. His concept of a 'general recursive function' is a natural extension of that in the case of natural numbers in a certain sense; however, it is rather what we want to call 'classical' and it differs from ours, even if the number of the variables is restricted to be finite.

In this paper, we shall introduce partial recursive functions as an extension of general recursive functions in the sense of Takeuti-Kino and give a formal system for them. In much of the symbolism, the notations and terminology, we follow S. C. Kleene [2] or Machover [4]. Let ω_γ be an arbitrary, but fixed, regular initial ordinal. Throughout this paper, by a function we shall always mean a function (or a functional) with a finite number of arguments ranging over ordinals $<\omega_\gamma$ (and with a finite number of function arguments) whose values are also ordinals $<\omega_\gamma$.

In § 1 we define formally calculable functions by establishing a formalism of function calculation. Roughly speaking, our system is obtained by adapting Machover's system (with infinitistic rules of formation and transformation)

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