Semisimple algebras over a commutative ring

Dedicated to Professor Y. Akizuki on his 60th birthday

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We intend in this paper to generalize the theory of semisimple algebras to the case where the domain of coefficients is a general commutative ring. In the classical case, a semisimple algebra A over a field K is characterized by one of the following three properties. 1) The direct sum of a finite number of simple algebras. 2) Vanishing of the radical. 3) Complete reducibility of every A-module. Each of these may be generalized in some manners to algebras over a commutative ring R. For instance, one may naturally call an algebra A over R to be simple if any two-sided ideal of A is of the form $\mathfrak{a}A$ where $\mathfrak{a}$ is an ideal of R. But, this definition seems to be too restrictive, since an extension of a Dedekind domain is not necessarily simple in this sense, even if it is unramified.

In this paper, we shall deal with the subject from the module-theoretical point of view 3). An algebra A over R will be called left semisimple if any extension of left A-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \quad \text{(exact)},$$

with finitely generated M_3, A-splits whenever it R-splits. In other words, A is left semisimple if every finitely generated left module is (A, R)-projective (Higman [9, 10], Hochschild [11]). We study in § 1 some aspects of relative homological algebra for later use even in somewhat more generalized formulations than exactly needed, not pursueing, however, deeper results for its own sake.

§ 2 deals with some basic properties of semisimple algebras. As in the classical case, separable algebras form the most important class of semisimple algebras. The theory of separable algebras over a ring is successfully developed by Azumaya [3] and Auslander and Goldman [2]. It turns out that the relations of the separability with the semisimplicity in the classical case remain hold in our general case, except that the question of separability of central semisimple algebras is left open. In § 3, we deal with the commuter relations of semisimple subalgebras of a central separable algebra.