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On Gaussian sums attached to the general linear groups over finite fields

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Introduction. Let $F(q)$ $(=F_q)$ be the finite field with q elements, q being a power of a prime number p. We denote by $M_n(F_q)$ the total matric ring of degree n over F_q, and by $GL(n,q)$ the group of regular elements of $M_n(F_q)$. To any irreducible representation ξ of $GL(n,q)$ by complex matrices, we can attach *Gaussian sums* $W(\xi,A)$ as follows. For every positive integer d, and for every $\alpha\in F(q^d)$, put

$$e_d[\alpha]=\exp\Big[\frac{2\pi\sqrt{-1}}{p}\,\mathrm{Tr}_{F(q^d)|F(p)}(\alpha)\Big].$$

Then, for every $A\in M_n(F_q)$, we define $W(\xi,A)$ by

$$W(\xi,A)=\sum_{X\in G(n,q)}\xi(X)e_1[\mathrm{tr}\,(AX)].$$

The matrices of this kind were investigated in E. Lamprecht [1] for the multicative groups of more general finite rings, but the explicit values of these matrices were not obtained.

The purpose of the present paper is to determine explicitly $W(\xi,A)$ for non-singular A and any irreducible representation ξ of $GL(n,q)$. To explain our result, first we note that, if $A\in GL(n,q)$,

$$W(\xi,A)=\xi(A)^{-1}W(\xi,1_n),$$

where 1_n denotes the identity element of $M_n(F_q)$. Moreover, we see easily that $W(\xi,1_n)$ is a scalar matrix. Then define a complex number $w(\xi)$ by

$$W(\xi,1_n)=w(\xi)\xi(1_n).$$

Fix once and for all an isomorphism θ of the multiplicative group of $F(q^{n!})$ into the multiplicative group of complex numbers. Further fix a generator ε of the multiplicative group of $F(q^{n!})$, and put, for every integer d such that $1\leqq d\leqq n$,

$$\varepsilon_d=\varepsilon^{\kappa},\qquad \kappa=\frac{q^{n!}-1}{q^d-1}.$$

Then ε_d is a generator of the multiplicative group of $F(q^d)$. For every irreducible polynomial g of degree d with coefficients in $F(q)$, we define the usual