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On the theory of reductive algebraic groups over a perfect field

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The purpose of this note is to prepare some basic theorems on reductive algebraic groups which will be used in our subsequent papers. The results obtained here seem more or less well-known to the mathematicians working in this field, but we found it convenient to resume them in a paper. The main idea in proving them is a systematic use of the notion of "Γ-fundamental system".

NOTATIONS AND TERMINOLOGY. In this paper, we consider exclusively affine (hence linear) algebraic groups defined over a *perfect* field k. For such a group G, G_k denotes the subgroup formed of all k-rational points in G. G° is the connected component of G containing the neutral element (except for the notation introduced in § 4). Since k is perfect, the words 'k-closed' and 'defined over k' are used quite synonymously. An isomorphism (resp. an isogeny) defined over k will be called briefly a k-isomorphism (resp. a k-isogeny). For any field k, k^* denotes the multiplicative group of all non-zero elements in k. $\mathbf{G}_m, \mathbf{G}_a$ are the multiplicative group of all non-zero elements in the universal domain and the additive group of the universal domain, respectively, considered as an algebraic group of dimension 1. For a subgroup H of an (abstract) group G, $N(H), Z(H)$ denote the normalizer and the centralizer of H in G, respectively. As usual, $\mathbf{Z}$ (resp. $\mathbf{Q}$) denotes the ring of rational integers (resp. the field of rational numbers). For any subset M of a module (resp. vector space over $\mathbf{Q}$), the symbol $M_{\mathbf{Z}}$ (resp. $M_{\mathbf{Q}}$) represents the submodule (resp. linear subspace over $\mathbf{Q}$) generated by M.

§ 1. Preliminaries.

1. Let k be a perfect field and G a connected (linear) algebraic group defined over k. It was proved by Rosenlicht [**9**] (cf. also [**5**], [**10**]) that the following four conditions on G are equivalent.

(T1) There exists a k-isomorphism from G into $T(n)$ ($=$the group of all upper triangular matrices of degree n).

(T2) G is solvable, and all characters of G (i.e. morphisms from G into $\mathbf{G}_m$)