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Univalent functions and non-convex domains

By Noriyuki SONE

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§ 1. Introduction.

The following theorem due to Noshiro [1] and Wolff [2] is well known.

THEOREM A. *If $f(z)$ is regular in a convex domain D and if $\Re f'(z)>0$ in D, then $f(z)$ is univalent in D.*

This theorem has been generalized by several authors from various points of view. Ozaki [3] proved the following:

THEOREM B. *Suppose that $g(z)$ is a convex univalent function in a domain D. If $f(z)$ is regular and $\Re\{e^{i\alpha}f'(z)/g'(z)\}>0$ (α real) in D, then $f(z)$ is univalent in D.*

Subsequently Kaplan [4] introduced a class of univalent mappings which he called 'close-to-convex'. These functions $f(z)$ defined in the unit disc, are characterized by an inequality of the form $\Re\{f'(z)/g'(z)\}>0$ where $g(z)$ is a convex univalent mapping of the unit disc. Obviously this characterization is a special case of the assumption of Theorem B. He further gave a characterization of these functions without reference to a convex function $g(z)$. It is essentially equivalent to the following Umezawa's criterion for univalence [5] i.e. the condition (i) of the following theorem, as Reade [8] points out.

THEOREM B'. *Let $w=f(z)$ be regular in a simply-connected closed domain D_z whose boundary Γ_z consists of a regular curve and suppose $f'(z)\neq 0$ on Γ_z. If there holds one of the following conditions:*

(i) *For arbitrary arcs C_z on Γ_z*

$$\int_{C_z} d\arg df(z)>-\pi \quad \textit{and} \quad \int_{\Gamma_z} d\arg df(z)=2\pi,$$

(ii) *For arbitrary arcs C_z on Γ_z* $\quad \int_{C_z} d\arg df(z)<3\pi,$

then $f(z)$ is univalent in D_z.

Recently Reade [9] proved Theorem C stated below using the following:

DEFINITION 1. Let φ be fixed, $0\leqq\varphi<\pi$. Then a domain D is said to be 'almost convex' if any distinct two points z_1, z_2 in D can be joined by a pair of straight line segments $\overline{z_1z_3}$, $\overline{z_3z_2}$ lying in D such that