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On predicates with constructive infinitely long expressions

Dedicated to Professor Y. Akizuki on his sixtieth birthday.

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In recent years the logic with infinitely long expressions has been considered and developed by Berkeley school. (For this see [1], [2], [7] and also cf. [6].) In this paper, we shall consider 'constructive' infinitely long expressions. In the following we shall give a language for the logic with infinitely long expressions and define a 'formula with constructive infinitely long expressions' as a formula with infinitely long expressions (sometimes called simply a formula) to which a so-called Gödel number is assigned. We shall show that the nesting number of a formula with constructive infinitely long expressions (see below) is less than Church-Kleene's ω_1 (Theorem 1). Moreover we shall establish a correspondence between formulas with constructive infinitely long expressions and predicates in Kleene's analytic hierarchy (cf. [4]). We shall prove that a formula $\mathfrak{A}$ with constructive infinitely long expressions is representable in the $\Sigma_{n+1}^1 \cap \Pi_{n+1}^1$-form, if the maximal number of quantifiers nested in $\mathfrak{A}$ is n (cf. $\mathrm{n}'(\mathfrak{A})$ defined below) and especially, $\mathfrak{A}$ is representable in Σ_n^1 or Π_n^1 if the outermost logical symbol of $\mathfrak{A}$ is $\exists$ or $\forall$ (Theorem 2). On the other hand, any predicate expressible in the n-function quantifier form is representable by a formula $\mathfrak{A}$ with constructive infinitely long expressions such that $\mathrm{n}'(\mathfrak{A}) = n$ (Theorem 3). We shall also prove that every hyperarithmetical formula is representable by a quantifier-free formula with constructive infinitely long expressions (Theorem 4).

0. In this paper we shall use the following language:

Individual constants $0, 1, 2, \cdots$;

Variables $v_0, v_1, \cdots, v_i, \cdots$ $(i < \omega)$;

The predicate $=$;

Logical symbols $\neg, \vee, \wedge, \exists, \forall$.

Prime formulas are of the form $i = j$, $i = v_n$, $v_m = j$ and $v_m = v_n$, where i and j are individual constants. Formulas are composed from prime formulas as follows:

0.1. If $\mathfrak{A}$ is a formula, then $\neg\mathfrak{A}$ is a formula.