J. Math. Soc. Japan
Vol. 15, No. 1, 1963

Projective modules over weakly noetherian rings

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(Received May 31, 1962)

Introduction. Let R be a commutative ring with a unit element. Then the space of prime ideals of R with the Zariski topology is called the *prime spectrum* of R which is denoted by spec(R), and the subspace of spec(R) of all maximal ideals of R is called the *maximal spectrum* of R which is denoted by m-spec(R). The dimension of such a space is the supremum of the lengths of chains of irreducible closed subsets. (See [**1**] and [**10**].) For brevity, we shall call a ring *weakly noetherian* if m-spec(R) satisfies the descending chain condition on closed subsets. Our main objective in this paper is to prove

THEOREM. *If R is a weakly noetherian ring and* dim(m-spec(R)) *is finite, then any projective R-module is a direct sum of finitely generated projective R-modules.*

From this we can easily deduce that, over a commutative indecomposable semilocal ring[1], any projective module is free[2].

Now let R be a commutative ring, M an R-module. Then M is called *faithfully flat* if M satisfies any one of the following equivalent conditions (see § 6.4 [5] p. 57):

(a) A sequence of R-modules $N' \to N \to N''$ is exact if and only if $M \otimes_R N' \to M \otimes_R N \to M \otimes_R N''$ is exact.

(b) M is flat and, for any R-module N, the relation $M \otimes_R N = (0)$ implies $N = (0)$.

(c) M is flat and, for any homomorphism $v: N \to N'$ of R-modules, the relation $1_M \otimes v = 0$ implies $v = 0$ where 1_M is the identity automorphism of M.

(d) M is flat and, for any maximal ideal $\mathfrak{m}$ of R, $\mathfrak{m}M \neq M$.

To prove the main theorem, we shall prove that, if R is an indecomposable weakly noetherian ring, any projective module ($\neq (0)$) is faithfully flat.

We shall always be dealing with rings with unit element and unitary modules. Further, unless the contrary is stated, "module" means "left module". Λ denotes a ring (not always commutative) and R denotes a commutative ring.

1) A ring is called indecomposable if it has no non-trivial idempotents. A commutative ring is called semilocal if the number of the maximal ideals is finite.

2) See [7].