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Classification of $SO(n)$-bundles over the quaternion projective plane

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Let Ω be the quaternion projective plane and let $SO(n)$ be the rotation group of $(n-1)$-sphere S^{n-1}. It is well known that the equivalence classes of $SO(n)$-bundles over Ω are 1-1 correspondence with the homotopy classes of maps of Ω into the classifying space B_n of $SO(n)$, therefore the classification of $SO(n)$-bundles over Ω reduces to the computation of the homotopy classes of maps $f:\Omega\to B_n$. Since the cases $n=1,2$ are trivial we are interested in cases $n\geqq 3$. We denote by $\Omega(n)$ the set of the equivalence classes of $SO(n)$-bundles over Ω.

We shall prove

THEOREM. *$\Omega(3)$ is in 1-1 correspondence with the pairs $(m, \mu(m)Z_2)$ such that $\frac{m(m-1)}{2}\equiv 0 \pmod{12}$, where $\mu(m)=0$ if m is even, $\mu(m)=1$ if m is odd.*

$\Omega(4)$ is in 1-1 correspondence with the triples $(m, l, \bar{\mu}_1 Z_2+\bar{\mu}_2 Z_2)$ such that $\frac{m(m-1)}{2}\equiv 0$, $\frac{l(l-1-2m)}{2}\equiv 0 \pmod{12}$, where $\bar{\mu}_1$, $\bar{\mu}_2$ are functions of m, l such that

$\bar{\mu}_1=\bar{\mu}_2=0$		if m, l are both even,
$\bar{\mu}_1=1,$	$\bar{\mu}_2=1$	if m is odd, l is even,
$\bar{\mu}_1=1,$	$\bar{\mu}_2=0$	if m is even, l is odd,
$\bar{\mu}_1=1,$	$\bar{\mu}_2=0$	if m, l are both odd.

If $n\geqq 5, n\neq 8$ $\Omega(n)$ is in 1-1 correspondence with the pair (r, s) of integers.

$\Omega(8)$ is in 1-1 correspondence with the triple (r, s, t) of integers.

The proof is given in the Section 1. In the Section 2 we shall consider characteristic classes of $SO(n)$-bundles over Ω.

1. Ω has the cell decomposition, $S^4\overset{\nu}{\cup} e^8$, where ν denotes the Hopf map: $S^7\to S^4$. Thus the above computation is equivalent to the computation of the homotopy classes of extensions of extendable maps: $S^4\to B_n$, over Ω. As is well-known, we have $\pi_4(B_3)\cong Z$, $\pi_4(B_4)\cong Z+Z$, $\pi_4(B_n)\cong Z$. We denote by $\bar{\alpha}_n$, $(n=3,4)$, $\bar{\beta}_n$, $(n\geqq 4)$ generators of $\pi_4(B_n)$. Let $\varDelta_n:\pi_k(B_n)\to\pi_{k-1}(SO(n))$ be the boundary homomorphism of the homotopy exact sequence of the universal bundle of $SO(n)$. Then we can take $\bar{\alpha}_n$ and $\bar{\beta}_n$ such that $\varDelta_4(\bar{\alpha}_4)=\alpha_3$, $\varDelta_4(\bar{\beta}_4)=\beta_3$,