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A note on predicates of ordinal numbers

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We shall assume the axiom of constructibility (cf. [1], [3]) throughout this paper. In [8], we considered the hierarchy of predicates of ordinal numbers in the first or the second number-class. In this paper we shall consider ordinal numbers of higher number-classes and we shall define the notions of primitive and general recursive functions of those ordinal numbers similarly as in [8] by introducing $\omega_1, \cdots, \omega_n$ (where ω_i is the initial ordinal number of the $i+2$ nd number class) as initial functions of primitive recursive functions of ordinal numbers in the $n+2$ nd class. (We shall simply say an ordinal number is in the n th class if it is in the m th number-class for some $m \leqq n$.) Then the arguments given in §§1-6 in [8] will be available with only slight modifications. The main result of this paper states:

A predicate in Kleene hierarchy of predicates with variables of types $\leqq n+1$ (for $n \geqq 1$) is in $\Sigma_1^{n+1} \frown \Pi_1^{n+1}$ if and only if it is expressible by a general recursive predicate of ordinal numbers in the $n+2$ nd class.

By the way we shall define the classical hierarchy (cf. [2]) of ordinal numbers in the third class and classically expressible ordinal numbers. We shall denote the least ordinal number not classically expressible by ω_1^* and show the analogous properties of ω_1^* to those of ω^*. It seems to suggest some analogies between the classical hierarchy of predicates with variables of type-2 and Kleene hierarchy of predicates with variables of type-1.

In the following we shall show how to extend the considerations in [8] to the third class and we shall only show the outline for the extension to higher number classes. Some acquaintance with [8] is assumed throughout this paper. We shall often cite definitions, propositions and theorems concerning with ordinal numbers in the $n+1$ st class ($n \geqq 2$) by putting the superscript n to the corresponding ones in [8].

§ 1. Primitive recursive² functions.

We say simply 'a is an ordinal number', if a is an ordinal number in the third class. We follow [8] for most of notions and notations on ordinal numbers and use ω_1 as the initial ordinal number in the third number class.

DEFINITION. A function is said to be *primitive recursive²*, if it can be