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A note on p-valent functions

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1. Introduction.

The set of close-to-convex univalent functions introduced by Kaplan [**2**] and Umezawa [**3**] contains many familiar univalent ones, for instance the starlike functions, the functions convex in one direction, the functions starlike with respect to symmetrical points [**5**], and the functions with derivative of positive real part in the unit circle. It, however, does not contain the spiral-like ones.

Recently a wider sufficient condition for univalence which includes the weakest sufficient condition for spiral-likeness has been given by Ogawa [**1**], and it has been extended to the case of p-valence at the same time. His main theorem for p-valence may be stated without loss of equivalency as follows.

THEOREM A. *Let* $f(z)=z^p+\cdots$ *be regular in* $|z|\leqq r$, *and let* $f(z)f'(z)\neq 0$ *for* $0<|z|\leqq r$. *If* $f(z)$ *satisfies the condition*

$$\int_C [d\arg df(z)+kd\arg f(z)]>-\pi$$

for all arcs C *on* $|z|=r$, *where* k *is a real constant such that* $k>-(1+1/2p)$, *then* $f(z)$ *is* p-*valent in* $|z|\leqq r$.

The purpose of this note is to extend or improve Theorem A and some of other results in his paper [**1**].

2. Fundamental results.

LEMMA 1. *Let* $f(z)=z^p+\cdots$, $\varphi(z)$ *be regular in* $|z|\leqq r$ *and* $|z|<+\infty$ *respectively, and let* $f'(z)\neq 0$ *for* $0<|z|\leqq r$. *If neither* $f(z)$ *nor* $\varphi'(\log f(z))$ *vanishes on* $|z|=r$ *and the number of valence of* $f(z)$ *in* $|z|\leqq r$ *is larger than* p, *then there exists at least one arc* C *on* $|z|=r$ *such that*

$$\int_C d\arg d\varphi(\log f(z))\leqq -\pi\,. \tag{2.1}$$

PROOF. As shown by Ogawa [**1**, p. 434], under our assumption on $f(z)$ there exists such a loop C_w on the image curve of $|z|=r$ under $w=f(z)$ as neither passes nor surrounds the origin and satisfies the inequality