J. Math. Soc. Japan
Vol. 14, No. 3, 1962

Closedness of some subgroups in linear algebraic groups

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(Received June 29, 1961)
(Revised Feb. 10, 1962)

Let M, N be closed subgroups of a linear algebraic group. It is mentioned in [1], that D. Hertzig proved that the commutator group $[M, N]$ is closed if M, N are normal. (A proof is given in [2] 3-04 Proposition 1. This fact brings about some simplification of Borel's arguments as noted in [1].) We shall give in this paper a necessary and sufficient condition for M, N to the effect that $[M, N]$ be closed, (Theorem 8 below,) from which the result of Hertzig easily follows (cf. [2], 3), and which will have also some interesting consequences. (Corollaries 9, 10, 11, below.)

In this paper we use the following conventions:
The subgroup generated by G_1, G_2 is denoted by $G_1 \vee G_2$, and the connected component of the identity of an algebraic group G is denoted by G_0.

The authors owe a great deal to M. Nagata, who gave us many useful suggestions. We offer our cordial thanks to him.

LEMMA 1. *Let G be an algebraic group and let $S_1, \cdots, S_m$ be its closed irreducible subsets. Let $f_\lambda(x_1, \cdots, x_m)(\lambda \in \Lambda)$ be words with $x_i \in S_i$, such that for suitable $(a_1^\lambda, \cdots, a_m^\lambda) \in S_1 \times \cdots \times S_m, f_\lambda(a_1^\lambda, \cdots, a_m^\lambda) = 1$ for each $\lambda \in \Lambda$. Then the subgroup H of G generated by $f_\lambda(x_1, \cdots, x_m)$, where $(x_1, \cdots, x_m)$ ranges over $S_1 \times \cdots \times S_m$ and λ ranges over Λ, is closed and connected.*

PROOF. For each $\lambda \in \Lambda$, let C_λ be the set of all $f_\lambda(x_1, \cdots, x_m)$ with $x_i \in S_i$. Then the set $C_{\lambda_1} \cdots C_{\lambda_t}$ of products $y_1 \cdots y_t$ $(y_i \in C_{\lambda_i})$ is the image of a rational map from $(S_1 \times \cdots \times S_m) \times \cdots \times (S_1 \times \cdots \times S_m)$ (t-ple product) into G, whence $C_{\lambda_1} \cdots C_{\lambda_t}$ is a thick set ('ensemble épais' cf. [1]), i.e. the closure $C(\lambda_1, \cdots, \lambda_t)$ of $C_{\lambda_1} \cdots C_{\lambda_t}$ is irreducible and $C_{\lambda_1} \cdots C_{\lambda_t}$ contains a non-empty open subset of $C(\lambda_1, \cdots, \lambda_t)$. Since $1 \in C_\lambda$, we see that $C_{\lambda_1} \cdots C_{\lambda_t} \subseteqq C_{\lambda_1} \cdots C_{\lambda_t} C_{\lambda_{t+1}}$, whence $C(\lambda_1, \cdots, \lambda_t) \subseteqq C(\lambda_1, \cdots, \lambda_t, \lambda_{t+1})$. By the fact that $C(\lambda_1, \cdots, \lambda_t)$ are irreducible subvarieties of G, we see that there is a $C(\lambda_1, \cdots, \lambda_t)$, say $C(\lambda_1, \cdots, \lambda_u)$, such that every $C(\lambda_1', \cdots, \lambda_s')$ is contained in $C(\lambda_1, \cdots, \lambda_u)$. ($C(\lambda_1, \cdots, \lambda_t)$ which has maximum dimension is a required one). Then $C_{\lambda_1} \cdots C_{\lambda_u} \subseteqq H \subseteqq C(\lambda_1, \cdots, \lambda_u)$. $C(\lambda_1, \cdots, \lambda_u)$ is the closure of H, hence is a group. Since H contains a non-empty open subset of $C(\lambda_1, \cdots, \lambda_u)$ (because $C_{\lambda_1} \cdots C_{\lambda_u}$ does), we see that $H = C(\lambda_1, \cdots, \lambda_u)$. This completes the proof.

PROPOSITION 2. *Let M and N be closed subgroups of an algebraic group G,*