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On hierarchies of predicates of ordinal numbers

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We shall assume the axiom of constructibility (' $V=L$ ', see Gödel [4]) throughout this paper.

In a former paper [17], we considered the class of ordinal numbers less than a certain cardinal number ($>\omega$) and defined the notion of semi-recursive and recursive functions. In this paper we shall consider the second number class and define the notion of primitive recursive functions and predicates by following Kleene's definition of natural numbers (§ 1). We shall also define the notion of recursive functions (sometimes called general recursive functions) analyzing Kleene's definition of natural numbers. The classes of both recursive functions defined in [17] and in this paper are seen to coincide (§ 2). We can construct a model of set theory in the primitive recursive predicates (§ 3). By restricting the usage of ' primitive recursion ' only to define functionsneccessary to construct the model, we obtain elementary functions (§ 4). Then a predicate is general recursive if and only if it is expressible in both forms consisting of a universal and existential quantifiers prefixed to elementary predicates (§ 5). The class of predicates expressible in a given form consisting of a fixed succession of one or more quantifiers prefixed to a predicate is the same whether the predicate is allowed to be general recursive or primitive recursive (or elementary). We shall prove the enumeration theorem, the normal form theorem and the hierarchy theorem for the predicates described above (§ 6). Moreover we shall show that for $k\geqq 1$ the predicate expressible in the k-quantifier form in our sense is expressible in the $k+1$-function-quantifier form in Kleene's sense and vice versa. (This would also follow from the results of Kuratowski [13] and Spector [14].) An analytic predicate is expressible in both 2-function-quantifier forms in Kleene's sense, if and only if it can be expressible as a (general) recursive predicate in our sense (§§ 7-8). We shall characterize hyperarithmetical predicates in our hierarchy of ordinal numbers (§ 9). Let us call an ordinal number to be recursively expressible if it is expressible by means of (general) recursive functions, 0 and ω, and denote the least ordinal number not recursively expressible as ω^*. We shall show the predicate $a<\omega^*$ is not (general) recursive, i. e. not expressible in both 2-function-quantifier forms in Kleene's sense, but is expressible in Σ_2^1-form.