J. Math. Soc. Japan
Vol. 14, No. 2, 1962

On stable processes with boundary conditions

By Shinzo WATANABE

(Received Aug. 31, 1961)

§ 0. Introduction.

Let $x_t(w)$, $t \geqq 0$, be the symmetric stable process with exponent α and I be the open interval $(-1, 1)$. For any right continuous path function $x_t(w)$ starting at some point $x \in I$, let $\sigma(w)$ be the first time $x_t(w)$ leaves I. The absorbing barrier stable process with exponent α is derived from $x_t(w)$ by killing it at time $\sigma(w)$. This process, which proves to be Markovian, was investigated by M. Kac [**9**] and J. Elliott [**3**]. Kac discovered the formal expression of the infinitesimal generator of the semi-group attached to this process and Elliott determined the domain of the generator in case $0 < \alpha < 1$. The first purpose of this paper is to determine this generator for every α $(0 < \alpha < 2)$, and this will be done in §§ 1-2.

In § 3 we shall compute the distribution of the first exit place x_σ and shall obtain the following results

$$P_x(x_\sigma \in [1, \infty)) = 2^{1-\alpha} \frac{\Gamma(\alpha)}{\left[\Gamma\left(\frac{\alpha}{2}\right)\right]^2} \int_{-1}^{x} (1-y^2)^{\frac{\alpha}{2}-1} dy$$

$$P_x(x_\sigma \in d\xi) = \frac{\sin \frac{\alpha\pi}{2}}{\pi} \left(\frac{1-x^2}{\xi^2-1}\right)^{\frac{\alpha}{2}} \frac{d\xi}{|\xi - x|}, \qquad |\xi| > 1.$$

These results have been obtained recently by H. Widom [**14**] in a somewhat different way. Our method consists in deriving the integro-differential equations governing these quantities and solving them.

In § 4 we shall determine the generator of the semi-group of the stable process on the space of continuous functions and shall also determine the generator of the absorbing barrier stable process on $I^- = (-\infty, 0)$.

Elliott [**2**] determined the most general boundary conditions by which the operator

$$\tilde{\Omega} u(x) = P \int_{-1}^{1} \frac{u'(y)}{y - x} dy$$

becomes a generator of a Markov process on $[-1, 1]$. In § 5 we extend this result to the case with general α. Our boundary conditions are obtained immediately from Feller's boundary conditions for the one-dimensional diffusion