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Convexity theorems for Fourier series

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In this paper we shall investigate some convexity theorems for Fourier series. This paper consists with three parts, each of which contains two main theorems (Theorems 1-6). These theorems together with Riesz's theorem (Lemma 5 in § 6) and Dixon-Ferrar's theorem (Lemma 2 in § 3) will constitute a complete system of convexity theorems in this direction, while the last two theorems are independent of Fourier series.

Let $\varphi(t)$ be an even function, integrable in $(0, \pi)$ in Lebesgue sense, periodic of period 2π, and let

$$\varphi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt ,$$

$$\Phi_0(t) = \varphi(t), \qquad \Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \varphi(u) du \qquad (\alpha > 0), \tag{0.1}$$

and more generally, for any integer $k \geqq 0$ and $0 < t \leqq \pi$,

$$\Phi_0^k(t) = t^k \varphi(t), \qquad \Phi_\alpha^k(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} u^k \varphi(u) du \qquad (\alpha > 0). \tag{0.2}$$

The Fourier series of $\varphi(t)$ at $t=0$ is $a_0/2 + a_1 + \cdots + a_n + \cdots$. The n-th (C, β) sum of this series is

$$s_n^\beta = A_n^\beta \frac{1}{2} a_0 + \sum_{\nu=1}^{n} A_{n-\nu}^\beta a_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{\beta-1} s_\nu \qquad (-\infty < \beta < \infty),$$

where $s_n = s_n^0$, and A_n^β is defined by the identity

$$(1-x)^{-\beta-1} = \sum_{n=0}^{\infty} A_n^\beta x^n \qquad (|x| < 1).$$

In particular, $s_n^{-1} = a_n \to 0$ as $n \to \infty$.

We understand that $t \to 0$ means $t > 0$ and $t \to 0$.

These notations will be used throughout this paper, except when it is stated otherwise.

Part I.

1. Theorems (1).

THEOREM 1. *Let* $0 \leqq \beta$, $-1 \leqq c$, $0 < \beta - b \leqq \gamma - c$ *and* $c - b < 1$. (I) *If*