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Approximation by reduced fractions

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1. Let $\{\delta(n)\}$ be a sequence of non-negative numbers. J. W. S. Cassels [1] proved that the set of real numbers x in $0 \leqq x < 1$ for which

$$\left| x - \frac{m}{n} \right| < \delta(n) \tag{1}$$

for infinitely many integers m, n has measure 0 or 1. R. J. Duffin and A. C. Schaeffer [2] had shown that for some sequences $\{\delta(n)\}$, this set has measure 1 while the set of x for which (1) holds for infinitely many relatively prime integers m, n has measure 0. Using an extension of Cassels' method, we will prove

THEOREM 1. *For each sequence of non-negative numbers* $\{\delta(n)\}$, *the set* $\mathcal{E}$ *of* x *in* $0 \leqq x < 1$ *for which*

$$\left| x - \frac{m}{n} \right| < \delta(n), \qquad (m, n) = 1 \tag{2}$$

for infinitely many m, n *has measure* 0 *or* 1.

We may suppose in the proof that $\delta(n) \to 0$. Otherwise each x satisfies (2) for infinitely many n. In fact, suppose that $n_1 < n_2 < \cdots$ is a sequence for which $\delta(n_\nu) \geqq \delta > 0$. For (2) to be satisfied with $n = n_\nu$ it is sufficient for there to exist an m prime to n_ν in the interval $| n_\nu x - m | < n_\nu \delta$. The existence of such an m, for all x and all large ν, follows from the following lemma.

LEMMA 1. *The length* L_n *of the longest interval of consecutive integers not prime to* n *satisfies* $L_n = o(n)$.

PROOF. Let $(m, n) > 1$ for $m_1 < m \leqq m_2$. Then

$$0 = \sum_{m_1 < m \leqq m_2} \sum_{d \mid (m,n)} \mu(d) = \sum_{d \mid n} \mu(d) \sum_{d \mid m, m_1 < m \leqq m_2} 1$$

$$= \sum_{d \mid n} \mu(d) \left(\left[\frac{m_2}{d} \right] - \left[\frac{m_1}{d} \right] \right) = (m_2 - m_1) \sum_{d \mid n} \frac{\mu(d)}{d} + O(d(n))$$

$$= (m_2 - m_1) \frac{\phi(n)}{n} + O(d(n)) .$$

Here $d(n)$ is the number of divisors of n. It is known that $d(n) = O(n^\varepsilon)$, and $n\phi(n)^{-1} = O(n^\varepsilon)$. Choosing m_1 and m_2 so that $m_2 - m_1 = L_n$, we have $L_n = o(n)$.