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Fractional powers of dissipative operators

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Introduction

The object of the present paper is to investigate the properties of the fractional powers A^{α} of linear operators A in a Hilbert space $\mathfrak{H}$, when $-A$ is closed and *maximal dissipative* in the sense of Phillips [**15, 16**]. $-A$ is said to be dissipative if $\mathrm{Re}(Au, u) \geqq 0$ for every $u \in \mathfrak{D}[A]$, and $-A$ is maximal dissipative if it has no proper dissipative extension. It is known (see [**15**]) that a closed, maximal dissipative operator is densely defined, that $-A$ is closed and maximal dissipative if and only if $-A^*$ is, and also if and only if $-A$ is the infinitesimal generator of a *contraction semi-group* $\{\exp(-tA)\}_{0<t<\infty}$, that is, $\| \exp(-tA) \| \leqq 1$.

Following a suggestion due to Friedrichs [**4**], we shall say that A is *accretive* if $-A$ is dissipative. In what follows we shall be concerned with accretive rather than with dissipative operators.

The fractional powers A^{α} can be defined in a natural way, at least for $0 \leqq \alpha \leqq 1$, if A is closed and maximal accretive, and A^{α} are again closed and maximal accretive. Such fractional powers have been defined for a more general class of linear operators in Banach spaces by several authors (see, among others, Balakrishnan [**1, 2**], Glushko and Krein [**5**], Kato [**9**], Krasnosel'skii and Pustylnik [**12**], Krasnosel'skii and Sobolevskii [**13**], Sobolevskii [**17**], Solomiak [**18**], Yosida [**19**]).

One of the important results to be proved in the present paper is that, if A is closed and maximal accretive, A^{α} and $A^{*\alpha}$ are *comparable* for $0 \leqq \alpha < 1/2$; by this we mean that A^{α} and $A^{*\alpha}$ have the same domain $\mathfrak{D}_{\alpha}$ and that the ratios $\| A^{*\alpha}u \| / \| A^{\alpha}u \|$ for $u \in \mathfrak{D}_{\alpha}$ are bounded from above and from below by positive constants. Another result is that A^{α} and $A^{*\alpha}$ have an *acute angle* for $0 \leqq \alpha < 1/2$; by this is meant that $\mathrm{Re}\,(A^{\alpha}u, A^{*\alpha}u) / \| A^{\alpha}u \| \, \| A^{*\alpha}u \|$ is bounded from below by a positive constant (see Sobolevskii [**17**]). These results are remarkable in view of the fact that nothing is assumed for the relationship between the domains of A and A^* themselves or for the angle between A and A^*.

It follows from these results that $H_{\alpha} = (A^{\alpha} + A^{*\alpha})/2$ is nonnegative self-adjoint and that it is comparable, and has acute angle, with both A^{α} and