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Axioms of infinity of set theory

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Mahlo, in his penetrating papers [5], [6], proposed an axiom of set theory, assuring, roughly speaking, the existence of sets with very large cardinals. Recently Lévy [3], [4] gave an elegant equivalent axiom, which he called the axiom of strong infinity. Mahlo's axiom can be also stated as follows in making use of the concept of the super-complete model introduced by Shepherdson [8]. (We remind that a set a is called super-complete if and only if

$$\forall x \forall y(x \in a \wedge (y \subseteqq x \vee y \in x) \vdash y \in a);$$

and that a set a is said to be a model of a set theory T, if and only if the theory holds, when all quantifiers of T are restricted on a.)

Let f be a function from sets to sets. A set a will be called a *fixed point* of f, if and only if

$$\forall x(x \in a \vdash f(x) \in a).$$

A class A will be called *dense*, if and only if for every function there exists a fixed point of this function which is an element of A. Then Mahlo's axiom means: The class of all the super-complete models of Bernays-Gödel's set theory is dense.

To go further, let us denote with M^0 the Bernay-Gödel's set theory. In adding Mahlo's axiom to M^0, we obtain a new set theory M^1, with 'much more set' than in M^0. More generally, a stronger set theory M^{i+1} is obtained in adding the axiom: The class of all the super-complete models of M^i is dense, to M^i, $i=0,1,2,\cdots$. These M^i will be called *Mahlo's set theories.*

Let $\tilde{\mathfrak{K}}(B)$ mean that B is the class of all the super-complete models of a Mahlo's set theory. $\mathfrak{K}(A)$ (for a class A) will mean

$$\exists B \exists x(\tilde{\mathfrak{K}}(B) \wedge \forall a(a \in B \wedge x \subseteqq a \wedge x \in a \vdash a \in A)).$$

Then we have easily

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