J. Math. Soc. Japan
Vol. 13, No. 2, 1961

On semi-hereditary rings

By Shizuo ENDO

(Received June 18, 1960)
(Revised Nov. 14, 1960)

§ 1. Introduction.

A ring R with unit element is called "*left* (*right*) *semi-hereditary*" according to [**2**] if any finitely generated left (right) ideal of R is projective.

The purpose of this paper is to determine completely the structure of commutative semi-hereditary rings. A. Hattori has recently given in [**6**] a homological characterization of Prüfer rings, i.e., semi-hereditary integral domains. This was generalized by M. Harada [**5**] to commutative rings whose total quotient rings are regular. The results of this paper will include those results of [**5**] and [**6**].

In § 3 we shall give a necessary and sufficient condition for a ring to be regular by using the quotient rings. Also we shall introduce a notion of quasi-regular rings and show some properties of them.

In § 4 we shall characterize semi-hereditary rings by using the quotient rings as follows: A ring R is semi-hereditary if and only if the total quotient ring K of R is regular and the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a valuation ring. Furthermore we shall introduce a notion of algebraic extensions of regular rings and show that the integral closure R' of a semi-hereditary ring R in any algebraic extension K' of the total quotient ring K of R is also semi-hereditary.

In § 5, we shall first prove that a local ring R is a valuation ring if and only if $\text{w.gl.dim}\, R \leqq 1$. Secondly we shall show, as a generalization of [**6**], Theorem 2, that a ring R with the total quotient ring K is semi-hereditary if and only if $\text{w.gl.dim}\, R \leqq 1$ and $\text{w.gl.dim}\, K = 0$, or if and only if any torsion-free R-module is flat.

§ 2. Notations and terminologies.

Throughout this paper a ring will mean a commutative ring with unit element 1. Our notations and terminologies are, in general, the same as in [**2**] but we shall make the following modifications.

A local ring will mean a (not always Noetherian) ring with only one