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On certain characteristic subgroups of a finite group

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Introduction.

One often asks whether a given property $\mathcal{P}$ is preserved by a given class of group-extensions (E), provided that $\mathcal{P}$ is possessed by all the kernels and the factor groups of (E). For example, the problem to find a group R, any extension H of which by a nilpotent group is likewise nilpotent, has a trivial answer. But if we allow as H only normal (or arbitrary, etc.) subgroups of a fixed group G, the problem becomes closely related to the structure of G, and the solutions of it may serve as a sort of measure relative to that property of groups. In this connection, a theorem of Gaschütz [**3**] is quite interesting, which states that the Frattini subgroup $\Phi(G)$ of a finite group G satisfies the condition of our problem. But $\Phi(G)$ is not in general maximal among the solutions, as the characteristic subgroup $\Delta(G)(\supset\Phi(G))$ introduced in that paper of Gaschütz is also one of solutions. Since larger solution is more interesting in such a problem, one naturally asks for the largest. Unfortunately the largest solution does not always exist. A standard method to make its substitute is to form the intersection of all maximal solutions, a procedure followed for example by Baer [**2**] to define the weak hypercenter. The nature of the present problem however allows us a different approach: We require of R^{σ} to possess the same property in G^{σ} as R in G for every homomorphism $\sigma: G\rightarrow G^{\sigma}$. Then we can prove that there necessarily exists the largest one among R's. The requirement is certainly satisfied by $\Phi(G)$ and $\Delta(G)$. Moreover, our method is favorable in hat it goes well with the induction-arguments.

We can treat similarly several problems of the same type (e. g. concerning abelian subgroups instead of nilpotent, etc.), and obtain thus a series of characteristic subgroups of a finite group. Some of them may be explicitly determined; for example, the hypercenter may be interpreted as the largest solution of the problme concerning the nilpotency and allowing as H any subgroup of G.

Notations

$|G|$ order of a finite group G.