On the Dirichlet problem for quasi-linear elliptic differential equations of the second order

By Kiyoshi AKô

§ 1. Introduction.

1. The purpose of this note is to discuss the Dirichlet problem for the general quasi-linear elliptic differential equation of the second order

(1) $\qquad a_{ij}(x, u, \operatorname{grad} u)\partial^2 u/\partial x_i \partial x_j + b_k(x, u, \operatorname{grad} u)\partial u/\partial x_k = f(x, u, \operatorname{grad} u)$[1]

with the bounded coefficients b_k and with the free term f which is bounded if the argument u is bounded[2].

2. In solving the Dirichlet problem of second order elliptic equations it is well-known that there are three kinds of important estimation for linear elliptic equations (see, e. g., L. Nirenberg [12]). One of the three is Schauder's interior estimates which will not be used in this note. Two others of the three concern the estimations of solutions belonging to $C^2(\bar{G})$ in the Banach spaces $C^{1,\alpha}(\bar{G})$ and $C^{2,\alpha}(\bar{G})$[3]. The latter, namely the estimation in $C^{2,\alpha}(\bar{G})$ first derived by J. Schauder ([13], [14]) and called by the name of Schauder's boundary estimates, is complete. But the estimation in $C^{1,\alpha}(\bar{G})$ is, even though considerably satisfactory by efforts of many eminent scholars (here we quote, in particular, L. Nirenberg [11] and H. O. Cordes [3]), still incomplete in the present situation.

The author is sorry to say that the results of the present note are considerably general but will become complete when the estimation in $C^{1,\alpha}(\bar{G})$ is completely accomplished. For this reason the author also hopes to make better the estimation in $C^{1,\alpha}(\bar{G})$ in the near future.

3. The results of this note are formulated in the sections 4, 6 and 7. There we shall give four theorems; namely, *the Main theorem, the envelope*

---

1) Here $x = (x_1, \cdots, x_n)$ and $\operatorname{grad} u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$. Throughout this note we use the summation convention for doubly appeared indices.

2) Actually we shall concern the equation (1) under more general assumptions. But for the sake of brevity we shall state the results of this note under rather simpler assumptions in the introduction.

3) See § 2 and Lemmas A, C of § 3.