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On Gödel's theorem

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Introduction

By applying precisely the arguments in Richard's paradox to a formal system $\boldsymbol{P}$ K. Gödel [2] proved that, if $\boldsymbol{P}$ satisfies certain conditions, then the following propositions hold.

PROPOSITION 1. *If $\boldsymbol{P}$ is consistent, then $\boldsymbol{P}$ is ω-incomplete.*

PROPOSITION 2. *If $\boldsymbol{P}$ is consistent, then the statement '$\boldsymbol{P}$ is consistent' is not provable in $\boldsymbol{P}$.*

It is well known that conditions which must be satisfied by $\boldsymbol{P}$ are satisfied by many formal systems, e. g. the system in Principia Mathematica and the system of Fraenkel-v. Neumann's axiomatic set theory. From the proposition 2 it is said that, if a system including the theory of natural numbers is wide enough, then the consistency proof of the system would be very hard.

However, we must notice that the concept of 'consistency' in metamathematics and that of 'consistency' used in Gödel's proposition 2 are not the same. In the proof of Gödel's proposition 2 Gödel formulated the statement 'a formal system $\boldsymbol{P}$ is consistent' in a form $\forall xC(x)$. By Gödel's proposition 1 even if formulas $C(1), C(2), C(3), \cdots$ are provable in $\boldsymbol{P}$, the formula $\forall xC(x)$ is not necessarily provable in $\boldsymbol{P}$. In order to prove in our proof-theory that the system is consistent it is sufficient to show that formulas $C(1), C(2), C(3), \cdots$ hold, and it is not necessary to show that $\forall xC(x)$ holds.

In §1 we give a formal system $\boldsymbol{P}$. Let $\forall xC(x)$ be a formula to formulate in $\boldsymbol{P}$ the proposition that $\boldsymbol{P}$ is consistent. In §2 we prove that $C(1), C(2), C(3), \cdots$ are provable in $\boldsymbol{P}$ and $\forall xC(x)$ is not provable in $\boldsymbol{P}$ if $\boldsymbol{P}$ is consistent.

In §3 and §4 we give a consistency proof of $\boldsymbol{P}$ in which the transfinite induction is not applied. Our proof is a modification of W. Ackermann's consistency-proof of $\boldsymbol{P}$ [1].

§1. The formal system $\boldsymbol{P}$.

To clarify the distinction between the strong form and the weak form of consistency formulated in a formal system, we give a formal system $\boldsymbol{P}$ as follows.

1. Symbols. $\boldsymbol{P}$ contains following fundamental symbols: the particular