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Note on the Kummer-Hilbert reciprocity law.

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1. Introduction.

Let p be an odd prime number, Q the field of rational p-adic numbers, ζ a fixed primitive p-th root of unity, and $k=Q(\zeta)$.

The classical Kummer-Hilbert reciprocity law was purely locally proved by K. Yamamoto [8] in the following form.[1]

Let $\mathfrak{p}$ be the prime ideal, and π an arbitrary prime element in k. By making use of the polynomial

$$\operatorname{Log}(1+x)=\sum_{i=1}^{p-1}\frac{(-1)^{i-1}}{i}x^{i},$$

we define differential quotients $l_{\pi}^{(i)}(\nu)$, which are determined modulo p, for a principal unit ν in k as follows:

$$\operatorname{Log}\nu\equiv\sum_{i=1}^{p-1}\frac{1}{i!}l_{\pi}^{(i)}(\nu)\,\pi^{i}\qquad(\mathfrak{p}^{p}).$$

Then it is necessary and sufficient for ν to be a norm of an element of $K=k(\sqrt[p]{\mu})$, where μ is a principal unit in k, that we have

$$\sum_{n=1}^{p-1}(-1)^{n-1}l_{\pi}^{(n)}(\nu)\,l_{\pi}^{(p-n)}(\mu)\equiv 0\quad(p).$$

Since the Lemmas 5, 5′ in [8], which are of importance in the local proof, contain an error, we shall make an attempt to obtain explicit formulas of general forms correcting [8], and therefrom we shall show that we may derive the above reciprocity law naturally. In the last section of this note we also obtain a formal generalization of the classical differential quotients of Kummer.

We exclude the case that μ is primary, i.e., K/k does not ramify, in which case we have $l_{\pi}^{(i)}(\mu)\equiv 0\ (p)$ for all i and the above proposition is evidently true.

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1) Cf. also Hilbert [4], Takagi [7], Artin and Hasse [1], Šafarevič [6], Kneser [5], Dwork [2].