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On polynomial approximation for strictly stationary processes.

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§ 1. Introduction and main results.

Consider a Brownian motion $B(t, \omega)$ and form a stochastic process

$$X(t, \omega)=\sum_{p} \sum_{i_1 \cdots i_p} a_{i_1 \cdots i_p} \prod_{\nu=1}^{p}(B(t+s_{i_\nu}, \omega)-B(t+u_{i_\nu}, \omega)) . \tag{1.1}$$

Here we shall call the process of the form (1.1) a polynomial process, since this process is a polynomial of the increments of $B(t, \omega)$. We see that a polynomial process is strictly stationary and that it is continuous in probability in the following sense:

$$\lim_{t \to s} P(|X(t, \omega)-X(s, \omega)|>\varepsilon)=0 \tag{1.2}$$

for $\varepsilon>0$ and any $s \in(-\infty, \infty)$.

It is obvious that an arbitrary strictly stationary process continuous in probability is not always a polynomial process, but we can approximate it in a certain sense by polynomial processes. The purpose of this paper is to prove this approximation theorem.

For this purpose, we shall introduce some topologies in the set of stochastic processes. The formal extension of the convergence in law for real random variables is as follows. A sequence of stochastic processes $X_n \equiv\{X_n(t, \omega),-\infty<t<\infty\}$ may be called to converge to the stochastic process $X \equiv\{X(t, \omega),-\infty<t<\infty\}$ in law if any joint distribution of X_n at a finite number of t-values converges to the corresponding one of X in Helly's sense. This definition is inadequate; in fact, even if $X_n \rightarrow X$ and if $X_{n,m} \rightarrow X_n$, we cannot always find a sequence X_{n,m_n} such that $X_{n,m_n} \rightarrow X$. Therefore we shall here introduce a neighborhood system $\{U(X, \varepsilon)\}$ which yields a convergence stronger than the convergence above.

DEFINITION 1. $U(X, \varepsilon)$ is the collection of all stochastic processes $Y \equiv \{Y(t, \omega),-\infty<t<\infty\}$ such that

$$|E e^{i \theta_1 X(t_1, \omega)+\cdots+i \theta_n X(t_n, \omega)}-E e^{i \theta_1 Y(t_1, \omega)+\cdots+i \theta_n Y(t_n, \omega)}|<\varepsilon$$

whenever n, $|\theta_j|$ and $|t_j|$ are all less than ε^{-1}.

Extending the convergence in probability for real random variables we shall say that X_n converges to X in probability if