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On the conformal mapping of nearly circular domains.

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1. Let us denote by C a closed Jordan curve on w-plane, contained in $1-\varepsilon\leqq|w|\leqq 1+\varepsilon$ for $0<\varepsilon<1$ and surrounding the origin, and denote by D the interior of C. When ε is sufficiently small, D is a so-called nearly circular domain. Let $w=F(z)$ be the function mapping the interior of the unit circle $|z|<1$ conformally onto D such that $F(0)=0$, $F'(0)>0$. The estimates of various quantities related to D or $F(z)$ in terms of ε have been given by various authors, recently by S. E. Warschawski [8], E. Specht [5], and Z. Nehari and V. Singh [4]. In [8] and [5], $d\arg F(e^{i\theta})/d\theta$ is estimated under some additional conditions for C. We treat, in this paper, the similar problems under somewhat different conditions, where C is not necessarily starlike with respect to the origin and there may be several angular points on it. Further we derive the inequalities concerning $|F'(e^{i\theta})|$, $\arg F(e^{i\theta})-\theta$, etc. We consider next about the expansion of $F(z)$ by ε. The results obtained there are possibly helpful to the numerical computation of $F(z)$.

2. We begin with several lemmas.

LEMMA 1. *Let $\varDelta$ be the sum of two open circular discs $|w|<1$ and $|w-a|<r$, where $0<r\leqq 1$ and $1-r<a<1+r$, and $e^{i\alpha}$, $e^{-i\alpha}$ $(0<\alpha<\pi/2)$ the intersections of those circumferences. Further we denote by $w=f(z)$ the function mapping $|z|<1$ conformally onto $\varDelta$ such that $f(0)=0$, $f'(0)>0$, and put $f(e^{i\beta})=e^{i\alpha}$, $f(e^{-i\beta})=e^{-i\alpha}$. Then $d\arg f(e^{i\theta})/d\theta$ for $-\beta<\theta<\beta$ attains its maximum at $\theta=0$.*

PROOF. The function $w=f(z)$ is represented explicitly by the composition of the functions

$$z=\frac{1+i\zeta\tan\beta/2}{1-i\zeta\tan\beta/2}, \tag{1}$$

$$w=\frac{\cos\dfrac{\alpha-\delta}{2}}{\cos\dfrac{\alpha+\delta}{2}}\,\frac{1+i\omega\tan\dfrac{\alpha-\delta}{2}}{1-i\omega\tan\dfrac{\alpha+\delta}{2}} \tag{2}$$

and

$$\frac{1+\omega}{1-\omega}=\left(\frac{1+\zeta}{1-\zeta}\right)^{1+\delta/\pi}, \tag{3}$$