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On certain properties of parametric curves.

Dedicated to Professor Z. Suetuna in celebration of his sexagenary birthday.

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Introduction.

We are mainly concerned with continuous parametric curves in the m-dimensional Euclidean space $\boldsymbol{R}^m$, where $m \geqq 2$, and intend to initiate a differential-geometric theory of such curves under general conditions. In principle, differentiability will not be imposed on the curves. We shall accordingly be naturally led to betake ourselves to methods of real function theory. Especially, certain properties of length of parametric curves will be indispensable for our purposes.

The most important quantities in classical differential geometry of curves are obviously curvature and torsion (besides arc length). However, these two are not capable of direct extension to our situation, inasmuch as they are local quantities involving differentiation. We shall therefore take another way and introduce a global quantity, called *bend*, certain of whose properties will constitute the main subject matter of the present paper. In fact, bend is closely related to curvature as we shall presently see in the next paragraph, and its theory is expected to be preparative to our further study. As regards torsion, it may fairly be said that we have obtained virtually no results as yet.

We define a parametric curve in $\boldsymbol{R}^m$ to be a mapping φ of a nonvoid set E of real numbers into $\boldsymbol{R}^m$. We shall regard $\boldsymbol{R}^m$ as a vector space whenever convenient. In the rest of the introduction we shall restrict ourselves for simplicity to curves defined on an interval. Let I be a closed interval. When φ is a regular $\boldsymbol{C}^2$ curve on I, classical theory applies, and we can consider the integral of the curvature of φ with respect to arc length, along the whole curve. We shall call this quantity, *integrated curvature* of φ. As is easily seen, this coincides with the length of the spheric representation of φ.

Returning to the general case we define, for every continuous curve φ on I, two quantities $\Theta(\varphi)$ and $\Omega(\varphi)$ as follows. We denote by $\Theta(\varphi)$ the lower