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Some properties of the Stone-Čech compactification.

By Hisahiro TAMANO

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In this note, we shall investigate some topological and uniform properties of Tychonoff space X (completely regular T_1-space) in connection with the properties of the Stone-Čech compactification βX.

The existence of compactification, the complete regularity and the uniformizability are equivalent each other, so that the Stone-Čech compactification may reasonably be expected to play an important role in the theory of uniform spaces. The consideration of uniformity $\mathcal{U}=\{V_\alpha\}$ in $\beta X\times\beta X$ leads us to consider the set $\boldsymbol{R}=\bigcap_\alpha \tilde{V}_\alpha$, where $\tilde{V}_\alpha$ denotes the interior of the closure of V_α taken in $\beta X\times\beta X$. The set $\boldsymbol{R}$ defined above will be called throughout as the radical of uniform space $(X, \mathcal{U})$. We shall show that the radical determines topologically the completion $\hat{X}$ of $(X, \mathcal{U})$. In fact, $\hat{X}$ is obtained as a quotient space $\bar{X}/\mathcal{R}$ (with the quotient topology), where $\bar{X}=\{p\in\beta X; (p, p)\in\boldsymbol{R}\}$ and $\mathcal{R}$ is the relation on $\bar{X}$ defined by the radical $\boldsymbol{R}$. The completeness will be characterized in terms of the radical as follows: $(X, \mathcal{U})$ is complete if and only if $\boldsymbol{R}=\Delta_X$. As a direct consequence of this, we shall obtain a necessary and sufficient condition for an entirely normal space to be topologically complete (Theorem 2.2). (We call the space X entirely normal if the family of all neighborhoods of the diagonal of $X\times X$ forms a uniformity for X.) The condition is stated as a property of points contained in $\beta X-X$ (points at infinity). A slightly stronger condition will be examined as well, and the relationship between entire normality and paracompactness will be made clear in a simple form (Theorem 2.3).

The idea to treat the completion of uniform space in connection with the compactification is due to H. Nakano [**11**]. We shall be concerned with the completion of uniform space in §3 and discuss some topological properties of the completion of uniform space in terms of the radical.

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