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Some remarks on Einstein spaces and spaces of constant curvature.

Dedicated to Professor Z. Suetuna on his 60th birthday.

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§ 1. Preliminaries.

The object of the present paper is to generalise some of recent results of André Avez [1]* to the case of non-compact Einstein spaces and to the case of spaces of constant curvature.

We shall here give notations and the formulas which will be used in the sequel.

Let M be an n dimensional Riemannian space of class C^4 with the fundamental metric tensor $g_{\mu\lambda}$ whose signature is not necessarily positive definite. We denote by ∇_μ the covariant differentiation with respect to the Christoffel symbols $\{{}_\mu{}^\kappa{}_\lambda\}$, by $K_{\nu\mu\lambda\kappa}$ the curvature tensor, by $K_{\mu\lambda}$ the Ricci tensor and by K the curvature scalar.

For an arbitrary skew-symmetric tensor field $w: w_{\lambda_1\lambda_2\cdots\lambda_p}$ of order p, we write

(1.1) $$(dw)_{\mu\lambda_1\lambda_2\cdots\lambda_p} = (p+1)\nabla_{[\mu}w_{\lambda_1\lambda_2\cdots\lambda_p]}$$

and

(1.2) $$(\delta w)_{\lambda_2\lambda_3\cdots\lambda_p} = \nabla_\mu w^\mu{}_{\lambda_2\lambda_3\cdots\lambda_p}.$$

Then the de Rham operator $\Delta = d\delta+\delta d$ applied to w gives [2]

$$(\Delta w)_{\lambda_1\lambda_2\cdots\lambda_p} = g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda_1\lambda_2\cdots\lambda_p}$$
$$-pK_{[\lambda_1}{}^\mu w_{|\mu|\lambda_2\cdots\lambda_p]} - \frac{p(p-1)}{2}K_{[\lambda_1\lambda_2}{}^{\nu\mu}w_{|\nu\mu|\lambda_3\cdots\lambda_p]}.$$

Especially, if w is a vector field,

(1.3) $$(\Delta w)_\lambda = g^{\nu\mu}\nabla_\nu\nabla_\mu w_\lambda - K_\lambda^\kappa w_\kappa$$

and if w is a skew-symmetric tensor field of order two,

$$(\Delta w)_{\lambda\kappa} = g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda\kappa} - 2K_{[\lambda}{}^\mu w_{|\mu|\kappa]} - K_{\lambda\kappa}{}^{\nu\mu}w_{\nu\mu}$$

or

(1.4) $$(\Delta w)_{\lambda\kappa} = g^{\nu\mu}\nabla_\nu\nabla_\mu w_{\lambda\kappa} - (2K_{[\lambda}{}^{[\nu}A_{\kappa]}^{\mu]} + K_{\lambda\kappa}{}^{\nu\mu})w_{\nu\mu}.$$

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* See the Bibliography at the end of the paper.