Journal of the Mathematical Society of Japan Vol. 12, No. 1, January, 1960

On some partition functions.

Dedicated to Professor Z. Suetuna on his completion of sixty years.

By Shô ISEKI

(Received July 30, 1959)

Introduction. Let $p_{\kappa}(n; a, M)$ be the number of partitions of a positive integer n into positive summands of the form $(Ml\pm a)^{\kappa}$ $(l=0,1,2,\cdots)$, where M, a and κ are integers satisfying $M\geqq 2$, $0<a<M$, $(a, M)=1$ and $\kappa\geqq 1$.

The first object of the present paper is to derive a suitable transformation formula for the generating function of $p_{\kappa}(n; a, M)$ and to determine the asymptotic behavior of the generating function in the neighborhood of its singularity at each rational point of the unit circle. A precise (not asymptotic) transformation equation will be obtained in § 1 of this paper.

Secondly, we shall give, in § 2, an asymptotic formula for the partition function $p_{\kappa}(n; a, M)$ for large values of n.

The special case $\kappa=1$ of our partition problem has been discussed in [3].

It should be noted that the case $M=2$ is equivalent to the case $M=4$, since we clearly have $p_{\kappa}(n; 1, 2)=p_{\kappa}(n; 1, 4)$. Therefore we may assume that $M\geqq 3$ in the sequel.

1. The transformation equation. The generating function of $p_{\kappa}(n; a, M)$ is given by

$$F_{\kappa}(x; a, M)=1+\sum_{n=1}^{\infty}p_{\kappa}(n; a, M)x^{n}=\prod_{\substack{\nu>0\\ \nu\equiv\pm a(M)}}(1-x^{\nu^{\kappa}})^{-1},$$

where x is a complex variable with $|x|<1$.

Now let h, k be coprime integers with $k\geqq 1$. We set

$(k, M)=D$ (the greatest common divisor of k and M),
$\{k, M\}=K$ (the least common multiple of k and M);

and put $k=k_{1}D$. Further we write

$$x=\exp(2\pi ih/k-2\pi z),$$

where z is a complex variable with $\Re(z)>0$. Define