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Conjugate classes of Cartan subalgebras in real semisimple Lie algebras.

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Introduction.

Cartan subalgebras play an important part in the theory of Lie algebras. Our main purpose in this paper is to find all Cartan subalgebras in real semisimple Lie algebras up to conjugacy under the adjoint groups or the full automorphism groups. The problem is very simple in complex semisimple Lie algebras, because all Cartan subalgebras in a Lie algebra over an algebraically closed field of chracteristic zero are mutually conjugate. This conjugacy theorem does not hold for a Lie algebra over a field which is not algebraically closed. For example, let $\mathfrak{g}=\mathfrak{sl}(2, R)$ be the Lie algebra of all 2×2 real matrices with trace zero, then $\mathfrak{g}$ has two Cartan subalgebras

$$\mathfrak{h}_1=\left\{\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix};\ t\in R\right\} \text{ and } \mathfrak{h}_2=\left\{\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix};\ \theta\in R\right\}.$$

$\mathfrak{h}_1$ and $\mathfrak{h}_2$ are not conjugate under an inner automorphism of $\mathfrak{g}$, because $\mathfrak{h}_1$ generates a non compact group

$$H_1=\left\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix};\ t\in R\right\},$$

while $\mathfrak{h}_2$ generates a compact group

$$H_2=\left\{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix};\ \theta\in R\right\}.$$

The conjugate classes of Cartan subalgebras in a Lie algebra over a general field of characteristic zero were first treated by N. Iwahori and I. Satake [8]. They proved the conjugacy of Cartan subalgebras for solvable Lie algebras. Later, the conjugate classes of Cartan subalgebras (or subgroups) attracted the attention of mathematicians in connection with the theory of unitary representations. I. M. Gelfand and M. I. Graev remarked that the existence of $[n/2]+1$ different conjugate classes of Cartan subgroup in $SL(n, R)$ is connected with the existence of $[n/2]+1$ different principal non degenerate series of irreducible unitary representations of $SL(n, R)$. Harish-Chandra, interested also in this phenomenon, proved that