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Some results on arithmetic functions.

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1. Introduction. Let $g(n)$ be an arithmetic function defined for all positive integers n. Following an idea of Yamamoto, we make correspond to $g(n)$ a linear operator $I_g(f)$ acting on the space of all functions $f(x)$ $(x\geqq 1)$, defined by

$$(I_g f)(x)=\sum_{n\leqq x}\frac{g(n)}{n}f\left(\frac{x}{n}\right). \tag{1}$$

The linear operators I_g were dealt with extensively in a previous paper ([**1**]), with the aid of a symbolic calculus introduced and used to approximate I_g for functions $f=f(\log x)$ which are polynomials in $\log x$.

Let $G(D)=\sum_{\nu=-p}^{\infty}g_\nu D^\nu$ be a power series in a symbol D with only a finite number of negative powers of D. The symbol D stands for the formal derivative $d/d\log x$. That is we set:

$$D^k\log^n x=\frac{n!}{(n-k)!}\log^{n-k}x \quad \text{for } n\geqq k$$

and all positive and non-positive values of k, $D^k\log^n x=0$ for $n<k$.

The notation $I_g=G(D)+O(\varphi_n)$ serves in [**1**] to denote that $I_g\log^n x-G(D)\log^n x=O(\varphi_n(x))$, where $\varphi_n(x)$, $n\geqq 0$, are non-negative functions. In a more explicit form, the last relation states that

$$I_g\log^n x-G(D)\log^n x=\sum_{\nu\leqq x}\frac{g(\nu)}{\nu}\log^n\frac{x}{\nu}-\sum_{i=-p}^{n}\frac{n!}{(n-i)!}g_i\log^{n-i}x=O(\varphi_n). \tag{2}$$

It is known that $I_gI_h=I_k$ where $k=g*h$ is the convolution of g and h, i.e. $k(n)=\sum_{d|n}g(d)h(n/d)$. Let $I_g=G(D)+O(\varphi_n)$ and $I_h=H(D)+O(\psi_n)$ then it was shown in [**1**, Theorem 4.1] that $I_gI_h=G(D)H(D)+O(\rho_n)$ and a certain bound for ρ_n was given, which was not symmetric in g and h; furthermore, an important drawback of that theorem was that $G(D)H(D)f$ had always to be computed as $G(D)[H(D)f]$ and not by the ordinary product of the power series $[G(D)H(D)]f$. This fact caused some complications in the computation in the proof of Theorem 9.1 of [**1**].