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On regular rings.

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Introduction

The term "*regular ring*" will be understood in this paper in the sense defined by Auslander-Buchsbaum [**2**]. It will mean namely a Noetherian ring R, such that the quotient ring $R_{\mathfrak{p}}$ of R with respect to any prime ideal $\mathfrak{p}$ of R is a regular local ring. A regular intergral domain will be simply called a regular domain. For example every Dedekind domain[1] is a one-dimensional regular domain. As is well known, the concept of regular local rings was introduced as a generalization of formal power-series rings with finite numbers of variables over fields, whereas a regular ring may be considered as a generalization of a polynomial ring with a finite number of variables over a field. In [**2**], as well as in Serre [**10**], an important characterization of regular local rings is given. (But in [**2**] most proofs are left out). The following theorem, given also in [**2**], [**10**] with homological methods and referred to as Theorem A in the following, is important for us.

THEOREM A. *If R is a regular local ring, then the quotient ring $R_{\mathfrak{p}}$ of R with respect to any prime ideal $\mathfrak{p}$ of R is also a regular local ring.*

According to this theorem, the definition of the regular ring can be restated as follows: *A Notherian ring R is called a regular ring if the quotient ring $R_{\mathfrak{m}}$ of R with respect to any maximal ideal $\mathfrak{m}$ of R is a regular local ring.*

In this paper, we shall start from our latter definition of the regular ring, and shall prove properties of regular rings using only purely ideal-theoretical methods. Among the results proved by homological methods, we presuppose only Theorem A above mentioned. Most of the results previously obtained, will be proved by simpler methods in the generalized form. We shall use, in this paper, the notations and terminology of Northcott [**8**]. Moreover, "ideal" will always mean a proper ideal, "ring" a commutative ring with unity e.

In § 1, we shall prove that every regular ring can be expressed as a direct sum of a finite number of regular domains.

1) Dedekind domain is an integral domain which satisfies Noether-Sono's condition.