Journal of the Mathematical Society of Japan Vol. 11, No. 2, April, 1959

Note on a B^*-algebra.

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(Received Sept. 15, 1958)
(Revised Dec. 12, 1958)

As is well-known I. Gelfand and M. Neumark [**1**] proved in 1943 that a Banach algebra over the complex number field with a unit 1 and an involution $*$ satisfying

(0.1) $x^{**}=x$,

(0.2) $(\alpha x)^{*}=\alpha^{*}x^{*}$ ($\alpha^{*}=$ the conjugate complex number of α),

(0.3) $(x+y)^{*}=x^{*}y^{*}$,

(0.4) $(xy)^{*}=y^{*}x^{*}$

is isometric and $*$-isomorphic to a C^*-algebra, (i. e., a uniformly closed self-adjoint algebra acting on a Hilbert space over the complex number field) if and only if it satisfies the following three conditions:

(0.5) $\|x^{*}x\|=\|x^{*}\|\|x\|$,

(0.6) $\|x^{*}\|=\|x\|$, and

(0.7) $1+x^{*}x$ has an inverse.

Also they conjectured that

(A) this fact holds without (0.7) and

(B) this fact holds without (0.6) (and (0.7)).

It was pointed out by I. Kaplansky that M. Fukamiya [**2**] gave implicitly an affirmative answer to the conjecture (A) (See J. A. Schatz' review [**3**] of [**2**]), and the assumption of existence of a unit was removed by I. Kaplansky and C. E. Rickart (cf. loc. cit.).

Their answer is very simple and stands on the following three facts:

(0.8) A B^*-algebra without a unit is isometrically and $*$-isomorphically imbeddible into a B^*-algebra with a unit (I. Kaplansky, C. E. Rickart).

(0.9) The set of non-zero spectra of xy (x, y being elements of a Banach algebra) coincides with that of yx.

(0.10) The set of hermitian elements of a B^*-algebra with a unit has a semi-ordering $h\geqq 0$ defined by $h=k^{2}$ (h, k being in the set) (M. Fukamiya [**2**], J. L. Kelley-R. L. Vaught [**5**]). (cf. I. Kaplansky [**4**] [**8**], M. Mimura [**6**]).

In this note, we shall give a direct proof of the theorem of I. Geland and M. Neumark by making no use of (0.7) in § 1. The present proof is not simple, for we make use neither of (0.7) nor of (0.8)—(0.10). In § 2, we shall give an affirmative answer to the conjecture (B).