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# On intermediate many-valued logics.

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There have been many reseaches on many-valued propositional logics. Rosser and Turquette [1], Dienes [2] and Church [3] investigated many-valued logical extensions of two-valued logic which have the analogous properties to classical logic. Łukasiewicz and Tarski [4] and Kleene [5] gave many-valued propositional logics which are not considered to be classical logic. Furthermore, the truth-tables given in [4] and [5] do not contain all formulas which are provable in intuitionistic propositional logic. In fact, $(A\supset \neg A)\supset \neg A$ which is provable intuitionistically does not always take the designated truth value in [4] and $A\supset A$ in [5] where $\supset$ and $\neg$ denote implication and negation respectively.

A treatment of many-valued propositional logics, in which every intuitionistically provable formula is true but not necessarily all classically provable formulas, viz. of intermediate many-valued logics in our terminology, was first achieved by Jaśkowski [6]. The purpose of this paper is to investigate details of intermediate many-valued logics.

A sufficient condition for a many-valued propositional logic to contain every intuitionistically provable propositional formula is given in § 1. Let $L_1,\cdots,L_n$ be arbitrary many-valued logics. We call $L_1,\cdots,L_n$ mutually independent, if for every distinct $i$ and $j$ there is a formula which is true in $L_i$ and not true in $L_j$. In § 2, it is proved that there are at least enumerably infinite mutually independent many-valued propositional logics.

In § 3 we construct a sequence of intermediate many-valued propositional logics in which every member is a sublogic of the preceding ones. This sequence is well-ordered and the ordinal number of the sequence is called the length of the sequence. It is proved that there is a sequence of intermediate many-valued propositional logics whose length is $\omega^{\omega^{\omega}}$. In § 4, special many-valued propositional logics $\mathfrak{R}_n$ and $\mathfrak{R}_\omega$ are discussed. The many-valued logics which can be reduced to $\mathfrak{R}_n$ is studied. Every provable formula in $\mathrm{LR}_n$ and $\mathrm{LP}_2$, special intermediate propositional logics in axiomatic stipulation (cf. Umezawa [8] and [9]), is true in $\mathfrak{R}_n$ and $\mathfrak{R}_\omega$ respectively.

In § 5 we extend the results in § 2 and § 3 to predicate calculus. Quantifiers $\forall$ and $\exists$ can be defined in the propositional logics which appear in the