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On a certain univalent mapping.

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§ 1. Univalent functions starlike with respect to symmetrical points.

Let $f(z)$ be regular in the unit circle, and suppose that for every r less than and sufficiently close to one and every ζ on $|z|=r$, the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z=\zeta$ as z traverses the circle $|z|=r$ in the positive direction, viz.

$$\Re \frac{zf'(z)}{f(z)-f(-\zeta)}>0 \qquad \text{for } z=\zeta,\ |\zeta|=r .$$

Then $f(z)$ is said to be starlike with respect to symmetrical points.

Obviously the class of functions univalent and starlike with respect to symmetrical points includes the classes of convex functions and odd functions starlike with respect to the origin.

THEOREM 1. *Let $f(z)=z+\cdots$ be regular in $|z|<1$. Then a necessary and sufficient condition for $f(z)$ to be univalent and starlike with respect to symmetrical points in $|z|<1$ is that*

$$\Re \frac{zf'(z)}{f(z)-f(-z)}>0, \quad |z|<1 . \tag{1.1}$$

PROOF. (1) Proof for necessity. We suppose that $f(z)$ is univalent and starlike with respect to symmetrical points. Then

$$f(z)-f(-z)\neq 0, \qquad 0<|z|<1 , \tag{1.2}$$

and

$$\Re\{zf'(z)/(f(z)-f(-z))\}>0, \quad |z|=r , \tag{1.3}$$

for every r less than and sufficiently close to one. From (1.2) the function $zf'(z)/(f(z)-f(-z))$ is regular in $|z|<1$, and therefore from (1.3) we have

$$\Re\{zf'(z)/(f(z)-f(-z))\}>0, \quad |z|\leqq r ,$$

by virtue of the minimum principle for harmonic functions. Hence (1.1) follows.

(2) Proof for sufficiency. We next suppose that (1.1) holds. Then $f(z)$ is evidently starlike with respect to symmetrical points, and therefore it is sufficient to show that $f(z)$ is univalent in $|z|<1$.

Substituting $-z$ for z in (1.1), we have