Journal of the Mathematical Society of Japan Vol. 11, No. 1, January, 1959

The conformal transformation on a space with parallel Ricci tensor.

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(Received Aug. 15, 1958)

Introduction

It is known that the conformal transformation group G of a space M with a Riemannian metric g coincides with the isometry group of M with some Riemannian metric g' conformally related to g, provided that the Weyl conformal tensor never vanishes on M [**4**]. Some of studies of conformal transformations, e. g., study of the group structure of conformal transformation groups, are therefore reduced to studies of isometries. So topics on conformal transformations may be limited to the case where M is conformally flat or to the relation between conformal transformations and properties of M which are not conformally invariant, e. g., the property to be symmetric, to which E. Cartan refered in his very first paper on "a remarkable class of Riemannian spaces". (It was proved in [**7**], [**11**] that a locally symmetric space does not admit an infinitesmal non-homothetic conformal transformation unless it is conformally flat.) In this paper we shall examine the relation between conformal transformations and the property that the Ricci tensor of M is parallel, and establish:

THEOREM. *Let g and g' be two complete Riemannian metrics on a manifold M $(2<\dim M=n)$, such that the Ricci tensor of each of them is parallel. If g and g' are conformally related, they are homothetically related or (some connected component of) M with the metric g (and with g' also) is isometric to the sphere.*

Two Riemannian metrics g and g' on the same manifold are by definition *conformally [homothetically] related* if there exists a scalar ϕ on M such that $g'=\phi g$ [and ϕ is a constant]. ϕ is called the *associated function.*

COROLLARY 1. *Let M, $2<n$, be a complete connected Riemannian space whose Ricci tensor is parallel. Then, if M admits a conformal transformation, one of the three cases occurs:* 1) *it is an isometry,* 2) *it is homothetic and M is isometric to the euclidean space,* 3) *M is isometric to the sphere.*

COROLLARY 2. *A connected symmetric space does not admit a non-homothetic conformal transformation if it is not isometric to the sphere.*

As special cases of the theorem we already know the following three theorems which are necessary for the proof of Theorem and Corollary 1.