Journal of the Mathematical Society of Japan Vol. 11, No. 1, January, 1959

Compact homogeneous spaces and the first Betti number.

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(Received April 25, 1958)

1. Introduction. The main purpose of this note is to prove:

THEOREM 1. *Let M be an n-dimensional homogeneous space G/H under a compact connected Lie group G. Then we have*

$$\dim S(p)+B_1=n\,,$$

where $S(p)$ is the orbit of an arbitrary point p in M under the maximal (connected) semi-simple subgroup S of G and B_1 denotes the first Betti number of M.

Note that H is not assumed to be connected. In the sequel we shall preserve these hypotheses and notations.

COROLLARY 1. *If G is semi-simple, then $B_1=0$* (T. Frankel [**3**]). The converse is not true (even if G is effective), but we have

COROLLARY 2. *If $B_1=0$, G contains a semi-simple subgroup which is transitive on M* (H. C. Wang [**10**]).

COROLLARY 3. *If $n\leqq B_1$, then M is homeomorphic to the torus and, furthermore if G is effective, G is an n-dimensional toral group* (D. Montgomery and H. Samelson [**6**] and A. Borel [**1**]).

COROLLARY 4. *Any finite covering space of M has the same first Betti number as M.*

In course of the proof of the above theorem, we shall establish:

THEOREM 2. *M admits a G-invariant Riemannian metric such that for a vector field u the following three conditions are equivalent:* 1) *u is parallel,* 2) *u is harmonic, and* 3) *u belongs to the center C^L of G^L of G and u is orthogonal to $S(p)$ at p.*

COROLLARY 5. *A vector field $u(\neq 0)$ on the homogeneous space M is parallel with respect to some G-invariant Riemannian metric if and only if u belongs to the centralizer of G^L in the Killing algebra of M with some G-invariant Riemannian metric and $u(p)$ is not tangent to $S(p)$.*

COROLLARY 6. *Let h be a vector field on M harmonic with respect to a G-invariant Riemannian metric g. Then h is parallel with respect to some G-invariant metric, if and only if h belongs to the Lie algebra K^L of a compact Lie transformation group K of M. If in particular h is Killing with respect to some metric, h is parallel with respect to some (other) metric.*

If a vector field u satisfies 1) in Theorem 2, clearly there exists, for any