Journal of the Mathematical Society of Japan Vol. 10, No. 4, October, 1958

Hyperbolic transfinite diameter and some theorems on analytic functions in an annulus.

By Tadao KUBO

(Received Nov. 15, 1957)
(Revised June 17, 1958)

1. Preliminaries.

Let a, b be two points in $|z|<1$, then the hyperbolic metric $[a, b]$ is defined by

$$[a, b]=\left|\frac{a-b}{1-\bar{a}b}\right|. \tag{1}$$

Let E be a bounded closed set, contained entirely in $|z|<1$, such that E and $|z|=1$ bound a connected domain D_0.

By introducing the hyperbolic metric (1) in $|z|<1$, Tsuji ([15], [16]) defined a potential of positive mass distribution on E and a hyperbolic transfinite diameter of E, and obtained some results analogous to those of Frostman [2] and de la Vallée-Poussin [17] in the theory of logarithmic potential and also to those of Pólya and Szegö [12] in the theory of transfinite diameter.

We summarize the results obtained by Tsuji as follows:

(i) Let $d\nu(a)\geqq 0$ be a positive mass distributed on E of total mass 1 and consider

$$I(\nu)=\iint_E \log\frac{1}{[a, b]}\,d\nu(a)d\nu(b), \qquad \nu(E)=1, \tag{2}$$

$$V=\inf_{\nu} I(\nu), \quad \infty\geqq V>0. \tag{3}$$

Then there exists $\mu\geqq 0$, such that

$$I(\mu)=\iint_E \log\frac{1}{[a, b]}\,d\mu(a)d\mu(b)=V, \quad \mu(E)=1. \tag{4}$$

(ii) For the potential of the mass distribution $d\mu(a)$ on E

$$u(z)=\int_E \log\frac{1}{[z, a]}\,d\mu(a)=\int_E \log\left|\frac{1-\bar{a}z}{z-a}\right| d\mu(a), \tag{5}$$

we have, similarly to the result of Frostman,

$$\sup_{|z|<1}\{u(z)\}=V \tag{6}$$

and

$$u(z)=V \quad \text{on } E, \tag{6'}$$