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On the local cross-sections in locally compact groups.

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Let G be a topological group, H a closed subgroup of G, p the natural map of G onto the left factor space G/H. e will denote the identity of G. These notations G, H, p, e will keep these meanings throughout the paper. (We can of course deal with the right factor space just as the left factor space. So we limit ourselves on the consideration of the left factor space. The terms like factor space, coset etc. without further qualification will always mean left factor space, left coset etc. in the following.) A continuous map f defined on a neighborhood U of any point in G/H with values in G such that $pf(x)=x$ for each $x\in U$, is called a *local cross-section of H in G* (cf. [7]). It is known that H has a local cross-section in G, if H is a compact Lie group [1] or if G is a locally compact finite-dimensional (separable metric) group [4]. In this paper, we shall prove these facts by actually constructing local cross-sections. These results will be thus proved by a unified method in a simpler way than in the literature and we shall obtain another sufficient condition for the existence of a local cross-section (see Theorem 2 below.). As an application, we obtain a simple proof of a theorem on the dimensions of factor spaces (Theorem 3).

1. The fundamental theorem.

We begin with the following lemma.

LEMMA 1. *H has a local cross-section in G, if there exists a compact subset W of G containing e such that*

1) *WH is a neighborhood of e in G,*
2) $W^{-1}W\cap H=\{e\}$.

The converse is also true if G is locally compact.

PROOF. Suppose that there exists a compact subset $W\ni e$ satisfying 1), 2). Put $e^*=p(e)$, $U=p(W)$, then U is a neighborhood of e^* by 1) and $p'=p|W$ is the one-to-one map of W onto U by 2). Since W is compact, p' is topological and the inverse map $f=p'^{-1}$ is a local cross-section. To prove the converse, we can suppose without loss of generality that a local cross-section f is defined on a compact neighborhood U of e^* such that $f(e^*)=e$. Put $W=f(U)$. W is