Journal of the Mathematical Society of Japan Vol. 10, No. 3, July, 1958

On functions starlike in one direction.

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(Received Nov. 19, 1957)

§ 1. Introduction.

Let a function

(1.1) $$f(z)=\sum_{n=1}^{\infty} a_n z^n \qquad (a_1=1)$$

be regular in the unit circle. It is well known that (a) if $f(z)$ is starlike with respect to the origin in the unit circle, all the partial sums of the form

(1.2) $$\sum_{n=1}^{m} a_n z^n \qquad (1 \leqq m \leqq \infty)$$

are starlike with respect to the origin for $|z|<1/4$ and convex for $|z|<1/8$, (b) if $f(z)$ is convex in the unit circle, all the partial sums of the form (1.2) are starlike with respect to the origin for $|z|<1/2$ and convex for $|z|<1/4$, and (c) these bounds are all sharp [1], [2].

In this paper we shall consider, in § 4, the partial sums of the form

(1.3) $$\sum_{n=0}^{m} a_{2n+1} z^{2n+1} \qquad (0 \leqq m \leqq \infty),$$

which consists of the odd terms, and we shall show the following: (1) If $f(z)$ is starlike with respect to the origin in the unit circle, then the partial sum $\sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$ is starlike with respect to the origin for $|z|<(3-2\sqrt{2})^{1/2}$ and convex for $|z|<(11-2\sqrt{30})^{1/2}$, and all the partial sums of the form (1.3) are starlike with respect to the origin for $|z|<1/3$ and convex for $|z|<1/3\sqrt{3}$. (2) If $f(z)$ is convex in the unit circle, then the partial sum $\sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$ is starlike with respect to the origin for $|z|<1$ and convex for $|z|<(3-2\sqrt{2})^{1/2}$, and all the partial sums of the form (1.3) are starlike with respect to the origin for $|z|<1/\sqrt{3}$ and convex for $|z|<1/3$. (3) These bounds are all sharp.

In §§ 2 and 3, we shall study k-fold symmetric functions starlike in the direction of one ray for general k and $k=2$ respectively, and in § 4 we shall prove the above statement by using the results of §§ 2 and 3.