Journal of the Mathematical Society of Japan Vol. 10, No. 2, April, 1958

Invariant tensors under the real representation of unitary group and their applications.

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(Received July 17, 1957)

N. H. Kuiper and K. Yano [3][1] determined all tensors of certain kinds interesting in differential geometry, which are invariant under the proper orthogonal group of the n-dimensional vector space. They also studied tensors invariant under the group of proper orthogonal transformations fixing a unit vector in the n-dimensional vector space and gave some applications of the results. The purpose of this paper is to determine all tensors of the types they studied, which are invariant under the real representation of unitary group. We obtain some theorems in differential geometry by applying the results.

1. Let C^n be the n-dimensional complex Cartesian space and R^{2n} be the $2n$-dimensional real Cartesian space. We assign to $(z^1,\cdots,z^n)$ of C^n $(x^1,\cdots,x^{2n})$ of R^{2n}, where $z^\alpha=x^\alpha+\sqrt{-1}\,x^{n+\alpha}$ [2]. Then to every linear transformation σ of C^n corresponds a linear transformation σ' of R^{2n}. If $A_1+\sqrt{-1}\,A_2$ with real matrices A_1, A_2 of degree n is the matrix of σ, then

$$\begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}$$

is the matrix of σ'. A real matrix A of degree $2n$ corresponds to a complex matrix of degree n in this way if and only if it commutes with J_n, $AJ_n=J_nA$, where

$$J_n=\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix},$$

E_n being the unit matrix of degree n. Let $(x^1,\cdots,x^{2n})$ and $(y,\cdots,y^{2n})$ of R^{2n} correspond to $(z^1,\cdots,z^n)$ and $(w^1,\cdots,w^n)$ of C^n respectively. Then we have

$$\sum_{\alpha=1}^{n}\bar{z}^\alpha w^\alpha=\sum_{i=1}^{2n}x^iy^i+\sqrt{-1}\sum_{\alpha=1}^{n}(x^\alpha y^{n+\alpha}-x^{n+\alpha}y^\alpha).$$

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1) Numbers in brackets refer to the bibliography at the end of the paper.

2) Throughout the paper Greek indices $\alpha,\beta,\cdots$ run over the range $1,2,\cdots,n$ and Latin indices $h,i,j,k,\cdots$ over the range $1,\cdots,2n$.