Journal of the Mathematical Society of Japan Vol. 10, No. 1, January, 1958

On the genus of the alternating knot, I.

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(Received Aug. 14, 1957)
(Revised Oct. 25, 1957)

F. Frankel and L. Pontrjagin [2] and H. Seifert [5] have given methods of construction of an orientable closed surface spanning a given knot i.e. having a given knot as a boundary. Seifert [5] has defined the genus $G(k)$ of the knot k as the minimum of the genera of orientable closed surfaces spanning k, whose existences are assured by [2] and [5]. Now let d be the degree of the Alexander polynomial of k. Seifert has proved that we have always

$$\frac{d}{2}\leqq G(k) \tag{1}$$

where the equality holds, if k is a torus knot, but there are also cases where the equality does not hold. (There are namely knots, whose Alexander polynomials are 1 and which are not equivalent to circles.)

In this paper, we shall show that the equality holds in (1) in certain classes of alternating knots (Theorem 1.1). For example, "alternierender Brezelknoten" of type $(p_1, p_2, \cdots, p_{2n+1})$, p_i being odd, i.e. alternating knots, whose projections have p_i crossing points on each arm and divide the plane into $\sum_{i=1}^{2n+1} p_i+2$ regions, of which $2n+2$ are "black", belong to these classes. It will be shown, at the same time, that for an alternating knot k of our classes, the orientable closed surface spanning k, whose genus is just equal to $G(k)$, is obtained by Seifert's construction.

§ 1. Main theorem.

Let k be a knot[1] and let K be an image of a regular projection[2] of k onto the plane E and let K be oriented by the orientation induced by that of k. Let K have n double points $c_1, c_2, \cdots, c_n$, called the *crossing points*. One of the two segments through a crossing point c_i passes under the other. It is called the *lower* segment at c_i and the other the *upper* segment. The

1) A knot means a polygonal simple closed (oriented) curve in Euclidean three dimensional space E^3.

2) See [3].