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Exact sequences in the Steenrod algebra.

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J. P. Serre [1] has proved that the cohomology algebra $H^*(Z_2; q, Z_2)$ of the Eilenberg-MacLane complex $K(Z_2, q)$ with Z_2 coefficients is a polynomial algebra generated by $\mathrm{Sq}^I(u_q)$, where u_q is the generator of $H^q(Z_2; q, Z_2)$ and I runs over the admissible sequences with excess $<q$, Sq^I being the iterated Steenrod squaring operations. He has proved thereby that $H^{n+q}(Z_2; q, Z_2)$ remains 'stable' for $q>n$, and put $A^n(Z_2, Z_2)=H^{n+q}(Z_2; q, Z_2)$. The graded algebra $\sum_{n=0}^{\infty} A^n(Z_2, Z_2)$ is denoted by $A^*(Z_2, Z_2)$ and is called the *Steenrod algebra* (Cf. Adem [2], [3]). Following Adem [2], we shall denote the generators of $A^*(Z_2, Z_2)$ with Sq^I instead of $\mathrm{Sq}^I(u_q)$. The multiplication between these generators is determined by Adem's relations (Adem [2], [3]).

$$\mathrm{Sq}^\alpha \mathrm{Sq}^\beta = \sum_{t=0}^{[\alpha/2]} \binom{\beta-t-1}{\alpha-2t} \mathrm{Sq}^{\alpha+\beta-t} \mathrm{Sq}^t \qquad \mathrm{mod}\ 2, \quad 0 \leqq \alpha < 2\beta . \tag{1}$$

Let I_0 be any fixed sequence of integers. We can define a homomorphism α'_{I_0} of $A^*(Z_2, Z_2)$ into itself by $\alpha'_{I_0}\mathrm{Sq}^I=\mathrm{Sq}^{I_0}\mathrm{Sq}^I$, and another homomorphism α''_{I_0} by $\alpha''_{I_0}\mathrm{Sq}^I=\mathrm{Sq}^I\mathrm{Sq}^{I_0}$. If M is a certain fixed submodule of $A^*(Z_2, Z_2)$, then $\mathrm{Sq}^I \to \alpha'_{I_0}\mathrm{Sq}^I$ mod M or $\alpha''_{I_0}\mathrm{Sq}^I$ mod M define respectively cohomology operations. These operations are of interest in view of topological applications. (Cf. Cartan [4], Serre [1]).

In this paper, we consider the operator α'_n defined by $\alpha'_n \mathrm{Sq}^I=\mathrm{Sq}^{2^n}\mathrm{Sq}^I$ $(n=0, 1, \cdots)$. We denote the module generated by the sums of the images of α'_i $(i=0, 1, \cdots, n)$ with M_n for $n \geqq 0$, and put $M_{-2}=M_{-1}=0$. Obviously we have $M_n \supset M_{n-1}$. We shall give explicitly the generators of M_n mod M_{n-1} (Theorem 1) and those of $A^*(Z_2, Z_2)$ mod M_n (Corollary of Theorem 1), and apply this to prove the following result. We can define α_{n+3} and β_{n+3} for $n \geqq -2$ so that the following diagram is commutative, where p_n is the natural homomorphism $A^*(Z_2, Z_2) \to A^*(Z_2, Z_2)/M_n$ for $n \geqq 0$, and $p_{-2}=p_{-1}=id$.

$$\begin{array}{ccccc} A^*(Z_2, Z_2) & \xrightarrow{\alpha'_{n+3}} & A^*(Z_2, Z_2) & \xrightarrow{\alpha'_{n+3}} & A^*(Z_2, Z_2) \\ \downarrow p_n & & \downarrow p_{n+1} & & \downarrow p_{n+2} \\ A^*(Z_2, Z_2)/M_n & \xrightarrow{\beta_{n+3}} & A^*(Z_2, Z_2)/M_{n+1} & \xrightarrow{\beta_{n+3}} & A^*(Z_2, Z_2)/M_{n+2} . \end{array}$$