Journal of the Mathematical Society of Japan Vol. 10, No. 1, January, 1958

A reciprocity law of the power residue symbol.

By Yoshiomi FURUTA

(Received Aug. 26, 1957)

Let l be a positive rational prime number and k be an algebraic number field of finite degree, containing a primitive l-th root ζ of unity. Denote by ζ_n a primitive l^n-th root of unity, and set $k_{(n)}=k(\zeta_n)$.

Let $\mathfrak{p}$ be a prime ideal of k prime to l, and α be an element of k prime to $\mathfrak{p}$. For these α and $\mathfrak{p}$, we define the symbol $\left[\frac{\alpha}{\mathfrak{p}}\right]_n$ inductively as follows.

For $n=0$, we set always $\left[\frac{\alpha}{\mathfrak{p}}\right]_n=1$.

For $n\geqq 1$, this symbol is defined only when we have

$$l^n \mid N\mathfrak{p}-1 \quad \text{and} \quad \left[\frac{\alpha}{\mathfrak{p}}\right]_{n-1}=1, \tag{1}$$

and, if that is so, we set

$$\left[\frac{\alpha}{\mathfrak{p}}\right]_n=\zeta^x \tag{2}$$

whenever we have

$$\alpha^{\frac{N\mathfrak{p}-1}{l^n}}\equiv\zeta^x \quad (\mathrm{mod}\ \mathfrak{p}). \tag{3}$$

Since every l-th root of unity is mutually incongruent modulo $\mathfrak{p}$, the value of $\left[\frac{\alpha}{\mathfrak{p}}\right]_n$ is uniquely determined in k by (2) and (3).

If $\mathfrak{m}$ is an ideal of k prime to α and to l with the prime ideal decomposition

$$\mathfrak{m}=\mathfrak{p}_1^{m_1}\cdots\mathfrak{p}_r^{m_r},$$

then we set

$$\left[\frac{\alpha}{\mathfrak{m}}\right]_n=\left[\frac{\alpha}{\mathfrak{p}_1}\right]_n^{m_1}\cdots\left[\frac{\alpha}{\mathfrak{p}_r}\right]_n^{m_r}.$$

The symbol $\left[\frac{\alpha}{\mathfrak{m}}\right]_n$ is considered to generalize the Diriclet's 4-th power residue symbol and, as we see in the latter half of §1, it is closely related to the "restricted Artin's symbol" in Rédei [4].

In Kuroda [3], a reciprocity law of the 4-th power residue symbol is given and, as an application, the decomposition law of rational primes in