Journal of the Mathematical Society of Japan Vol. 9, No. 4, October, 1957

The Gauss-Bonnet Theorem for V-manifolds.

By Ichiro SATAKE

(Received Aug. 30, 1957)

Introduction.

The purpose of this paper is to generalize the Gauss-Bonnet theorem (established by Allendoerfer and Weil [1]) to the case of V-manifolds. The notion of a V-manifold has been introduced by the author in a previous short paper [7].

The outline of this paper is as follows. In § 1 we shall give fundamental concepts concerning V-manifolds and V-bundles. Our principal idea is the following: while an ordinary manifold can be considered as an inverse injective limit of Euclidean spaces, a V-manifold is considered as that of Euclidean spaces allowing a finite group of automorphisms. More precisely speaking, let $\{\tilde{U}_\alpha\}_{\alpha\in A}$ be a system of Euclidean spaces (of the same dimension), A being a directed system such that for any $\alpha, \beta\in A$ there exists a $\gamma\in A$ with $\gamma\leqq\alpha$, $\gamma\leqq\beta$ and assume that for any $\alpha, \beta\in A$, $\alpha\leqq\beta$ we have 'an injection' $\lambda_{\beta\alpha}$ from $\tilde{U}_\alpha$ into $\tilde{U}_\beta$ such that we have $\lambda_{\gamma\alpha}=\lambda_{\gamma\beta}\circ\lambda_{\beta\alpha}$ for $\alpha\leqq\beta\leqq\gamma$. Then the injective limit $M=\bigcup_\alpha \varphi_\alpha(\tilde{U}_\alpha)$ of the inverse system $\{\tilde{U}_\alpha, \lambda_{\beta\alpha}\}$, φ_α denoting the canonical injection $\tilde{U}_\alpha\to M$, is a manifold. ($\varphi_\alpha(\tilde{U}_\alpha)$ being homeomorphic to $\tilde{U}_\alpha$, M is locally homeomorphic to a Euclidean space.) Now a slight modification of the above definition will lead to a V-manifold. Namely replacing the word 'an injection' by 'a finite number of injections' we obtain a V-manifold $M=\bigcup \varphi_\alpha(\tilde{U}_\alpha)$, which is locally homeomorphic to $\varphi_\alpha(\tilde{U}_\alpha)\approx G_\alpha\backslash\tilde{U}_\alpha$, G_α being a finite group of automorphisms of $\tilde{U}_\alpha$, composed of $\lambda_{\alpha\alpha}$. Similar considerations can be applied also to the definition of V-bundles. Namely a V-bundle B can be considered as an injective limit (allowing a finite group of automorphisms) of an (inverse) system of direct products of a Euclidean space $\tilde{U}_\alpha$ and a fixed manifold F (called fibre) with respect to injections of a special form. As examples of V-bundles the notions of tangent vectors and differential forms will be introduced at the end of this section.