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Note on Random Riemann Sum.

By Makiko NISIO

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§ 1. Let $f\in L_p(a, b)$ and consider the Riemann sum $S_n\equiv S_n(f; t_1,\cdots, t_n)$ for random division points t_i. It was the idea of P. Lévy [1] to define 'generalized integrals' as 'limits' of such Riemann sums. S. Takahashi [2] proved that S_n converges to $\int_0^1 f(t)dt$ with probability 1, for $p>1$ and in probability for $p=1$, if the t_i, $1\leqq i<\infty$, are independent with the uniform distribution on (0,1).

We shall now consider the case $p=1$, $a=0$, $b=\infty$, taking, for each $n\geqq 1$, division-points t_i^n with probability ndt in $(t, t+dt)$ and mutually independently in non-overlapping time-intervals. Rigorously speaking, we adopt the jumping times of the Poisson process with parameter n as the division-points. We then let n tend to ∞.

In § 2 we shall prove that if $f\in L_1(0, \infty)$, S_n converges to the Lebesgue integral of f over $(0, \infty)$ in probability as n tends to infinity and in § 3 that if $f\in L_1(0, \infty)\cap L_2(0, \infty)$, the subsequence S_{2^n} converges to the same value with probability one.

It should be noticed that our way of picking the division-points by means of Poisson process has made much easier the analytical treatment compared with the case treated by Takahashi [2].

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§ 2. Let (Ω, B, P) be the probability space on which we define Poisson processes $\{P_n(t, \omega),\ t\in(0, \infty)\}$, n being the mean value of $P_n(1, \omega)$; it makes no difference here whether these processes are independent or dependent. Let $t_i^n(\omega)$ be the i-th jumping point of the Poisson process $P_n(t, \omega)$.

In this note we shall often use the following well-known fact.

LEMMA. $\{t_{i+1}^n(\omega)-t_i^n(\omega)\}$, $i=0, 1, 2,\cdots$ $(t_0^n(\omega)\equiv 0)$ *are independent random variables with the following probability law*:

$$P\{t_{i+1}^n(\omega)-t_i^n(\omega)<t\}=1-e^{-nt} \quad (t\geqq 0), \tag{2.1}$$

and so we have