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Ordinal diagrams.

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In his paper [2] on the consistency-proof of the theory of natural numbers, G. Gentzen assigned to every proof-figure an ordinal number. In modifying his method, we may do this as follows:

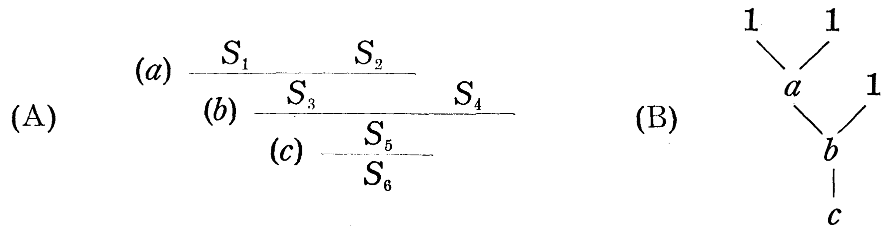

Fig. 1

Suppose, to fix our idea, a proof-figure (A) (in Fig. 1) is given. This is composed of beginning sequences S_1, S_2, S_4 and inferences (*a*), (*b*), (*c*). To the inferences: weakening, contraction and exchange, we assign the value 0; to a cut of degree n, the value n; to an induction of degree n, the value $n+1$; and the value 1 to all other inferences. We denote the values of inferences (*a*), (*b*), (*c*) by a, b, c respectively. We replace the beginning sequences by 1, and draw the figure (B) according to the form of the proof-figure (A).

If we consider $\overset{\alpha\ \beta}{\underset{a}{\vee}}$ and $\overset{\alpha}{\underset{a}{|}}$ (α, β being ordinal numbers and a a non-negative integer) as operations defining ordinal numbers (to be defined properly, see below), then the figure like (B) represents itself an ordinal number. This may be called 'Gentzen's number' for the proof-figure (A). Although this is not the same ordinal number as assigned to (A) by Gentzen himself, we can accomplish the consistency-proof of the theory of natural numbers just as in [2], in proving that this 'Gentzen's number' is diminished by the reduction of the proof-figure.

The operations $\overset{\alpha\ \beta}{\underset{a}{\vee}}$ and $\overset{\alpha}{\underset{a}{|}}$ can be described by Ackermann's