Journal of the Mathematical Society of Japan Vol. 9, No. 4, October, 1957

On Prüfer rings.

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(Received March 25, 1957)

To find a necessary and sufficient condition for an integral domain Λ to satisfy the following condition (C): (C) *If A and B are torsion-free Λ-modules, then $A\otimes_{\Lambda}B$ is also a torsion-free Λ-module.* This is a problem recently proposed by M. Nagata.[1]

We know, following J. Dieudonné,[2] that (C) is satisfied by any Dedekind ring, and more generally by any Prüfer ring, as is shown by H. Cartan and S. Eilenberg in their recent publication.[3] In this paper, we shall prove conversely that a ring satisfying (C) is necessarily a Prüfer ring (Theorem 2). This will solve the above problem completely, and at the same time yield a characterization of Prüfer rings.[4]

Let Λ denote an integral domain (with an identity). Instead of $A\otimes_{\Lambda}B$, $\mathrm{Tor}_n^{\Lambda}(A, B)$, $\mathrm{Hom}_{\Lambda}(A, B)$, $\mathrm{Ext}_{\Lambda}^n(A, B)$, we shall use simplified notations $A\otimes B$, $\mathrm{Tor}_n(A, B)$, $\mathrm{Hom}(A, B)$, $\mathrm{Ext}^n(A, B)$, A and B being Λ-modules. (See *HA*, for the definition of these *functors*).

LEMMA 1. *For a finitely (Λ-) generated torsion-free Λ-module A, there exists a free Λ-module F on finite basis containing A and such that the residue class module F/A is a torsion module.*

PROOF. We have only to modify the proof of *HA*, Prop. VII. 2.4: Let Q be the quotient field of Λ, then A is a submodule of $Q\otimes A$, and a system of Λ-generators $\{a_1,\cdots, a_r\}$ is also a system of Q-generators of the vector space $Q\otimes A$ over Q. Hence the set $\{a_1\cdots, a_r\}$ contains a Q-basis of $Q\otimes A$, say $\{a_1,\cdots, a_s\}$. If

$$a_i=\sum_{j=1}^{s} q_{ij}a_j, \qquad i=1,\cdots, r, \qquad q_{ij}\in Q,$$

1) Sûgaku, vol. 6.1 (July, 1954), Problem 6.1.13.

2) J. Dieudonné, *Sur les produits tensoriels*, Ann. de l'Ecole Norm. Sup. LXIV (1947), pp. 101-117. Théorème 3.

3) H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press (1956). Prop. VII. 4.5. In the following we shall refer to this book by *HA*.

4) The author published this result already in Sûgaku, vol. 8. (1957).