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On finite-dimensional perturbations of self-adjoint operators.

By Tosio KATO

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§1. Introduction. The purpose of the present paper is to show that a perturbation of finite rank does not change the main structure of a self-adjoint operator in a sense to be specified below[1], and to deduce certain asymptotic relationships between the one-parameter continuous groups generated by the unperturbed and perturbed self-adjoint operators.

Let $\mathfrak{H}$ be a (not necessarily separable) Hilbert space, H_0 a (not necessarily bounded) self-adjoint operator in $\mathfrak{H}$ and V a self-adjoint operator with finite rank m. Then $H_1=H_0+V$ is also self-adjoint with domain $\mathfrak{D}$ identical with that of H_0. We assert that H_1 and H_0 are unitarily equivalent to each other except for separable, singular parts with multiplicities not exceeding m. A more precise expression is given by

THEOREM 1. *Let H_0, H_1 be as above. Then there exist two subspaces*[2] $\mathfrak{M}_0$, $\mathfrak{M}_1$ *with respective projections P_0, P_1 and a subspace* $\mathfrak{M}_{01}\subset\mathfrak{M}_0\cap\mathfrak{M}_1$ *with the following properties.*

1) $\mathfrak{M}_{01}$ *reduces both H_0 and H_1; the parts of H_0 and H_1 in $\mathfrak{M}_{01}$ are identical.*

2) $\mathfrak{H}\ominus\mathfrak{M}_{01}$ *is (and hence $\mathfrak{H}\ominus\mathfrak{M}_0$ and $\mathfrak{H}\ominus\mathfrak{M}_1$ are a fortiori) separable; the parts of H_0 and H_1 in $\mathfrak{H}\ominus\mathfrak{M}_{01}$ have spectra with multiplicities not exceeding m.*

3) $\mathfrak{M}_0$ *and* $\mathfrak{M}_1$ *reduce H_0 resp. H_1; the parts of H_0 and H_1 in $\mathfrak{M}_0$ resp. $\mathfrak{M}_1$ are unitarily equivalent to each other.*

4) *The parts of H_0 and H_1 in $\mathfrak{H}\ominus\mathfrak{M}_0$ resp. $\mathfrak{H}\ominus\mathfrak{M}_1$ are singular.*

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1) In his Technical Report 17 "On a problem of Hermann Weyl in the theory of singular Sturm Liouville equations," N. Aronszajn states that he and W. F. Donoghue have obtained results similar to ours. Also M. Rosenblum announces in the abstract 99 in Bull. Amer. Math. Soc. 62 (1956) p. 30 results closely related to ours. (Added in proof) see end of paper.

2) We mean by a subspace a closed linear manifold of $\mathfrak{H}$.