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On Umezawa's criteria for univalence.

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1. In an interesting paper of recent date, Umezawa [5] obtained some new criteria that a function analytic in a certain domain should be univalent there. Those criteria all involve the change in direction of the tangent vector to the image of the boundary.

In this note we extend slightly some of Umezawa's results, and we give what we believe are simpler proofs yielding slightly more precise results. We use a device introduced by Umezawa, and a result due to Kaplan and Umezawa, to show that the function $\int_0^z e^{-\zeta^2}\,d\zeta$ is univalent for $|z|<1.51$; this improves upon estimates due to Nehari [2] and Rogozin [4].

We shall make use of the results obtained by Kaplan [1] in a recent paper in which he introduced univalent close-to-convex functions.

2. The following result is Umezawa's fundamental lemma [5; p. 213]. Our proof avoids Umezawa's geometric argument and shows that Umezawa's result is equivalent to Kaplan's fundamental result [1; p. 173].

THEOREM I. *Let $f(z)$ be analytic inside and on the simple closed analytic curve Γ, and let $f'(z)$ have no zeros on Γ. If*

$$\int_\Gamma d\arg df(z)=2\pi\,, \tag{1}$$

and if for all arcs C on Γ we have

$$\int_C d\arg df(z)>-\pi\,, \tag{2}$$

then $f(z)$ is univalent inside Γ, and the image of Γ a simple close-to-convex curve.

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